美国数学会经典影印系列



Topics in Optimal Transportation

Second Edition

最优输运理论专题

第二版

Cédric Villani

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出版者的话

近年来，我国的科学技术取得了长足进步，特别是在数学等自然科学基础领域不断涌现出一流的研究成果。与此同时，国内的科研队伍与国外的交流合作也越来越密切，越来越多的科研工作者可以熟练地阅读英文文献，并在国际顶级期刊发表英文学术文章，在国外出版社出版英文学术著作。

然而，在国内阅读海外原版英文图书仍不是非常便捷。一方面，这些原版图书主要集中在科技、教育比较发达的大中城市的大型综合图书馆以及科研院所的资料室中，普通读者借阅不甚容易；另一方面，原版书价格昂贵，动辄上百美元，购买也很不方便。这极大地限制了科技工作者对于国外先进科学技术知识的获取，间接阻碍了我国科技的发展。

高等教育出版社本着植根教育、弘扬学术的宗旨服务我国广大科技和教育工作者，同美国数学会（American Mathematical Society）合作，在征求海内外众多专家学者意见的基础上，精选该学会近年出版的数十种专业著作，组织出版了“美国数学会经典影印系列”丛书。美国数学会创建于 1888 年，是国际上极具影响力的专业学术组织，目前拥有近 30000 会员和 580 余个机构成员，出版图书 3500 多种，冯·诺依曼、莱夫谢茨、陶哲轩等世界级数学大家都是其作者。本影印系列涵盖了代数、几何、分析、方程、拓扑、概率、动力系统等所有主要数学分支以及新近发展的数学主题。

我们希望这套书的出版，能够对国内的科研工作者、教育工作者以及青年学生起到重要的学术引领作用，也希望今后能有更多的海外优秀英文著作被介绍到中国。

高 等 教 育 出 版 社

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To my optimal son, Neven

Preface of the Second Edition

Time has passed since the first edition of this book. My son, who was born during the writing of the manuscript, is now entering high school. As for the theory of optimal transport, I used to think of her as a young adult, but I now realize that in those days she was just a teenager. And she did grow up spectacularly since then, being enriched by many developments.

To begin with, the connection to non-Euclidean geometry, which was only hinted at in those days, has blossomed into an ambitious, fast developing theory. This led to major progress, not only in optimal transportation but also in synthetic geometry, through a program heralded by John Lott, Karl-Theodor Sturm and myself. It also boosted nonsmooth gradient flow theory under the expert hands of Luigi Ambrosio, Nicola Gigli, Giuseppe Savaré, and many others — a book on this subject was born shortly after mine, it looked like the final word but in fact it was just a step in a broader picture. And as a side product, through the skills of the Italian school, this direction of research also led to fundamental advances in metric analysis.

Further, the connection to dynamical systems and the Aubry–Mather theory was enormously pushed through the contributions of Patrick Bernard, Albert Fathi, Wilfrid Gangbo, and an illuminating reinterpretation of the older work of John Mather. This new light had a deep influence on further presentation of the subject.

Also, the theory of fully nonlinear partial differential equations was revisited with fracas as Xiaonan Ma, Neil Trudinger, Xiu-Ja Wang and Grégoire Loeper opened the door to the regularity theory of optimal transportation in

curved geometry, and essentially solved what I presented at the end of Chapter 3 as the most important open problem in the field. An ensuing revival of cut locus theory was just one of the associated amazing developments.

At the time of the first edition, I was expressing hope in the rise of certain numerical schemes, but in the end faster algorithms were developed in other ways, for instance by Quentin Mérigot and Marco Cuturi.

More generally, every feature of the theory has been revised, extended and generalized in nearly any possible way. Even more importantly, a crowd of young and talented researchers, expert in those subjects, has emerged – among many, let me quote François Bolley, Vincent Calvez, Fabio Cavalletti, Thierry Champion, Luigi De Pascale, Matthias Erbar, Max Fathi, Alessio Figalli, Nicola Gigli, Nathaël Gozlan, Young-Heon Kim, Kazumasa Kuwada, Grégoire Loeper, Jan Maas, Francesco Maggi, Emanuel Milman, Andrea Mondino, Aaron Naber, Aldo Pratelli, Tapio Rajala, Max Von Renesse, Ludovic Rifford, Filippo Santambrogio... It may be vain, but it has been an immense pride and joy to think that I participated in making this community emerge — by the writing of this book in the first place, but also by writing the next one, by lecturing on the subject all around the world, and by organizing events such as the Oberwolfach Seminar which I held in 2002 with Luigi Ambrosio.

In spite of all these developments, the present book has aged well and is still used regularly as a source of working seminars or as an introduction to the subject. Part of this longevity is certainly due to the informal and enthusiastic writing style, maybe typical of a young mathematician, and which I feel I can never recover. It is also probably due to the fact that my second book on the subject, *Optimal Transport, old and new* (Springer, 2008) has served as a reference textbook — bigger, more general, more geometric — but has not replaced the present course as an introduction to the field. For these reasons, I have decided to make the revision as modest as could be, limiting it mainly to the correction of mistakes and a few mandatory comments, and basically keeping the numbering of sections and statements unchanged. And likewise, I abandoned my longstanding desire to add a chapter on numerical simulations, preferring to leave that task to others.

Mistakes in the first edition were not so many, but not so few either. Many colleagues have helped me chase them, in particular Scott Armstrong, Eitan Bachmat, Patrick Cattiaux, Ulrik Fjordholm, Bo'az Klartag, Elliott Lieb, Sylvia Serfaty, Michel Valadier, Marcus Wunsch and Xiling Zhang. It is a pleasure to thank them all, with particularly warm feelings to Ulrik and Sylvia for the detailed reporting which came out of their working seminars.

With the passing of time it became clearer that my 1999 stay in Georgia Tech, Atlanta, which led to the writing of this course, had a huge impact

on my mathematician's life; so it is only appropriate that I thank again Wilfrid Gangbo, Eric Carlen and Michael Loss for organizing it. As far as my work in optimal transportation is concerned, additional thanks are due to my research collaborators, most notably Felix Otto, Luigi Ambrosio and John Lott; to my colleagues in Lyon, in particular Albert Fathi and Étienne Ghys; and to my students, of course.

Last but not least, let me evoke the memory of Ralph Sizer, who was the copy-editor of the first edition. The work which he did on the manuscript was outstanding, by and large. A few years later, I tried to get his advice for another project, and was shocked to learn about his untimely passing away. Although I never met Sizer, our long exchanges of mails made me feel like I knew him. Behind those ASCII characters and handwritten scribblings lie human emotions, and the joy of working together.

I shall conclude this foreword with comments on the presentation. I have left notation and conventions unchanged from the first edition, rather than updating them in accordance with some of my later writings. In particular, in this book,

- I use the word "transportation" rather than "transport";
- I denote by $T\#\mu$ the push-forward of the measure μ by the mapping T; while alternative conventions would be $T_{\#}\mu$ (which I used in *Optimal transport, old and new*) or $T_*\mu$, or $T\mu$;
- I use the full axiom of choice in a few places, most notably Section 1.3, while I completely banned its use from more recent writings;
- I use the spelling Aleksandrov rather than Alexandrov;
- I express the Kantorovich duality with either a pair of dual semi-convex functions, or a pair of dual semi-concave functions, while the convention which I used in more recent writings is rather in the form of one semi-convex and one semi-concave functions (facilitating the connection to Mather's theory).

As a final remark, the terminology "Wasserstein distance" (Chapter 7) has been objected to, in particular because those distances were first put forward by Leonid Kantorovich himself, at least in some cases. Still, I stuck to the use of Wasserstein, partly because it has become popular in the field, partly because so many key concepts are already named after Kantorovich.

I hope that readers will enjoy these notes as much as I enjoyed discovering the subject and writing about it, more than fifteen years ago.

Cédric Villani
Paris & Lyon, July 2015

Preface of the First Edition

This set of notes grew from a graduate course that I taught at Georgia Tech, in Atlanta, during the fall of 1999, on the invitation of Wilfrid Gangbo. It is a great pleasure for me to thank Georgia Tech for its hospitality, and all the faculty members and students who attended this course, for their interest and their curiosity. Among them, I wish to express my particular gratitude to Eric Carlen, Laci Erdös, Michael Loss, and Andrzej Swiech. It was Eric and Michael who first suggested that I make a book out of the lecture notes intended for the students.

Three years passed by before I was able to complete these notes; of course, I took into account as much as I could of the mathematical progress made during those years.

Optimal mass transportation was born in France in 1781, with a very famous paper by Gaspard Monge, *Mémoire sur la théorie des déblais et des remblais*. Since then, it has become a classical subject in probability theory, economics and optimization. Very recently it gained extreme popularity, because many researchers in different areas of mathematics understood that this topic was strongly linked to their subject. Again, one can give a precise birthdate for this revival: the 1987 note by Yann Brenier, *Décomposition polaire et réarrangement des champs de vecteurs*. This paper paved the way towards a beautiful interplay between partial differential equations, fluid mechanics, geometry, probability theory and functional analysis, which has developed over the last ten years, through the contributions of a number of authors, with optimal transportation problems as a common denominator.

These notes are definitely not intended to be exhaustive, and should rather be seen as an introduction to the subject. Their reading can be complemented by some of the reference texts which have appeared recently. In particular, I should mention the two volumes of *Mass transportation problems*, by Rachev and Rüschendorf, which depict many applications of Monge-Kantorovich distances to various problems, together with the classical theory of the optimal transportation problem in a very abstract setting; the survey by Evans, which can also be considered as an introduction to the subject, and describes several applications of the L^1 theory (i.e., when the cost function is a distance) which I did not cover in these notes; the extremely clear lecture notes by Ambrosio, centered on the L^1 theory from the point of view of calculus of variations; and also the lecture notes by Urbas, which are a marvelous reference for the regularity theory of the Monge-Ampère equation arising in mass transportation. Also recommended is a very pedagogical and rather complete article recently written by Ambrosio and Pratelli, and focused on the L^1 theory, from which I extracted many remarks and examples.

The present volume does not go too deeply into some of the aspects which are very well treated in the above-mentioned references: in particular, the L^1 theory is just sketched, and so is the regularity theory developed by Caffarelli and by Urbas. Several topics are hardly mentioned, or not at all: the application of mass transportation to the problem of shape optimization, as developed by Bouchitté and Buttazzo; the fascinating semi-geostrophic system in meteorology, whose links with optimal transportation are now understood thanks to the amazing work of Cullen, Purser and collaborators; or applications to image processing, developed by Tannenbaum and his group. On the other hand, I hope that this text is a good elementary reference source for such topics as displacement interpolation and its applications to functional inequalities with a geometrical content, or the differential viewpoint of Otto, which has proven so successful in various contexts (like the study of rates of equilibration for certain dissipative equations). I have tried to keep proofs as simple as possible throughout the book, keeping in mind that they should be understandable by non-expert students. I have also stated many results without proofs, either to convey a better intuition, or to give an account of recent research in the field. In the end, these notes are intended to serve both as a course, and as a survey.

Though the literature on the Monge-Kantorovich problem is enormous, I did not want the bibliography to become gigantic, and therefore I did *not* try to give complete lists of references. Many authors who did valuable work on optimal transportation problems (Abdellaoui, Cuesta-Albertos, Dall'Aglio, Kellerer, Matrán, Tuero-Díaz, and many others) are not even cited within

the text; I apologize for that. Much more complete lists of references on the Monge-Kantorovich problem can be found in Gangbo and McCann [**151**], and especially in Rachev and Rüschendorf [**225**]. On the other hand, I did not hesitate to give references for subjects whose relation to the optimal transportation problem is not necessarily immediate, whenever I felt that this could give the reader some insights in related fields.

At first I did not intend to consider the optimal mass transportation problem in a very general framework. But a graduate course that I taught in the fall of 2001 on the mean-field limit in statistical physics, made me realize the practical importance of handling mass transportation on infinite-dimensional spaces such as the Wiener space, or the space of probability measures on some phase space. Tools like the Kantorovich duality, or the metric properties induced by optimal transportation, happen to be very useful in such contexts — as was understood long ago by people doing research in mathematical statistics. This is why in Chapters 1 and 7 I have treated those topics under quite general assumptions, in a context of Polish spaces (which is not the most general setting that one could imagine, but which is sufficient for all the applications I am used to). Almost all the rest of the notes deals with finite-dimensional spaces. Let me mention that several researchers, in particular Üstünel and F.-Y. Wang, are currently working to extend some of the geometrical results described below to an infinite-dimensional setting allowing for the Wiener space.

A more precise overview of the contents of this book is given at the end of the Introduction, after a precise statement of the problem. I shall also summarize at the beginning the main notation used in the text; to avoid devastating confusion, note carefully the definition of a "small set" in $\mathbb{R}^n$, as a set of Hausdorff dimension at most $n-1$.

As the reader should understand, the subject is still very vivid and likely to get into new developments in the next years. Among topics which are still waiting for progress, let me only mention the numerical methods for computing optimal transportations. At the time of this writing, some noticeable progress seems to have been done on this subject by Tannenbaum and his coworkers. Even though these beautiful new schemes seem extremely promising, they need confirmation from the mathematical point of view, which is one reason why I skipped this topic (the other reason being my lack of competence). Some related results can be found in [**163**].

Also I wish to emphasize that optimal mass transportation, besides its own intrinsic interest, sometimes appears as a surprisingly effective *tool* in problems which do not a priori seem to have any relation to it. For this reason I think that getting at least superficially acquainted with it is a wise

investment for any student in probability, analysis or partial differential equations.

This book owes a lot to many people. I was lucky enough to learn the subject of optimal mass transportation directly from two of those researchers who have most contributed to turn it into a fascinating area: Yann Brenier and Felix Otto; it is a pleasure here to express my enormous gratitude to them. I first encountered optimal transportation in Tanaka's work about the Boltzmann equation, and my curiosity about it increased dramatically from discussions with Yann; but it was only after hearing a beautiful and enthusiastic lecture given in Paris by Craig Evans, that I made up my mind to study the subject thoroughly. My involvement in the study of functional inequalities related to mass transportation was partly triggered by interactions with Michel Ledoux, whose influence is gratefully acknowledged. The present manuscript profited a lot from numerous discussions with Luigi Ambrosio, Eric Carlen, Dario Cordero-Erausquin, Wilfrid Gangbo and Robert McCann. Both Robert and Luigi taught the material of this book, made many suggestions and pointed out numerous misprints and mistakes in the first version of these notes. The most serious one concerned the "proof" of Theorem 1.3, as given in the first version of these notes; the gap was fixed thanks to the kind help of Luigi again, and of Bernd Kirchheim, with the final result of an improved statement. Some of my students at the École normale supérieure also spotted and repaired a gap in the proof of Theorem 2.18. François Bolley, Jean-François Coulombel and Maxime Hauray should be thanked for the time they spent hunting for mistakes and misprints in various parts of the book, and testing many of the exercises and problems. Richard Dudley was kind enough to give a quick but thorough look at Chapters 1 and 7. Chapter 4 would not have existed without the explanations which I received from Luis Caffarelli and Andrzej Swiech. Most of the material in Chapter 6 was taught to me by Franck Barthe. Chapter 8 was reshaped by the exchanges which I had with Luigi Ambrosio, Nicola Gigli and Étienne Ghys during the last stages of preparation of the manuscript. Finally, Mike Cullen corrected some mistakes in the presentation of the physical model in Problem 9 of Chapter 10.

Cédric Villani
Lyon, January 2003

Notation

The identity map on an arbitrary space will be denoted by Id. Whenever X is a set, we write $1_X(x) = 1$ if $x \in X$, $1_X(x) = 0$ otherwise. The complement of a set A will be denoted by A^c.

Throughout the text, whenever we write $\mathbb{R}^n$ the dimension n is an arbitrary integer $n \geq 1$. Whenever A is a Lebesgue-measurable subset of $\mathbb{R}^n$, its n-dimensional Lebesgue measure will be denoted by $|A|$. This should not be confused with the Euclidean norm of a vector $x \in \mathbb{R}^n$, which will also be denoted by $|x|$. Whenever $x, y \in \mathbb{R}^n$ we write $x \cdot y = \langle x, y \rangle = \sum_{i=1}^n x_i y_i$.

Given some abstract measure space X, we shall denote by $P(X)$ the set of all probability measures on X, and by $M(X)$ the set of all finite signed measures on X (i.e. precisely the vector space generated by $P(X)$). The space $M(X)$ is equipped with the norm of total variation, $\|\mu\|_{TV} = \inf\{\mu_+[X] + \mu_-[X]\}$, where the infimum is taken over all nonnegative measures μ_+, μ_- such that $\mu = \mu_+ - \mu_-$. The infimum is obtained when μ_+ and μ_- are singular to each other, in which case $\mu = \mu_+ - \mu_-$ is said to be the Hahn decomposition of μ. Of course, if ν is a nonnegative measure and f a measurable map, then $\|f\|_{L^1(d\nu)} = \|f\nu\|_{TV}$. From Chapter 1 on, we shall only work in topological spaces, equipped with their Borel σ-algebra; so $P(X)$ will be the set of Borel probability measures. We shall sometimes write $w*\text{–}P(X)$ for $P(X)$ equipped with the weak topology.

The Dirac mass at a point x will be denoted by δ_x: $\delta_x[A] = 1$ if $x \in A$, 0 otherwise.

If a particular measure μ on X is singled out, for $p \in [1, \infty)$ we shall denote by $L^p(X)$ or $L^p(d\mu)$ the Lebesgue space of order p for the reference

measure μ, with the usual identification of functions which coincide almost everywhere. Whenever $p \geq 1$, we shall denote by p' its conjugate exponent:

$$\frac{1}{p} + \frac{1}{p'} = 1.$$

Whenever T is a map from a measure space X, equipped with a measure μ, to an arbitrary space Y, we denote by $T\#\mu$ the image measure (or push-forward) of μ by T. Explicitly, $(T\#\mu)[B] = \mu[T^{-1}(B)]$, where $T^{-1}(B) = \{x \in X;\ T(x) \in B\}$. The set of all $T : X \to X$ such that $T\#\mu = \mu$ will be denoted by $S(X)$. We shall always use push-forward in this sense: when we write $T\#f = g$, where f and g are nonnegative functions, this means that the measure having density f is pushed foward to the measure having density g (usually the reference measure will be the Lebesgue measure).

If X is a topological space, then it will be equipped with its Borel σ-algebra. We shall denote by $C(X)$ the space of continuous functions on X; by $C_b(X)$ the space of bounded continuous functions on X; and by $C_0(X)$ the space of continuous functions on X going to 0 at infinity. Sometimes these notations will be replaced by $C(X;\mathbb{R})$, $C_b(X;\mathbb{R})$, $C_0(X;\mathbb{R})$. The space $C_b(X)$ comes with a natural norm, $\|u\|_\infty = \sup_X |u|$. Whenever $A \subset X$, we denote by $\mathrm{Int}(A)$ the largest open set contained in A, and by $\overline{A}$ the smallest closed set containing A. We set $\partial A = \overline{A} \setminus \mathrm{Int}(A)$. By definition, the support of a measure μ on X will be the smallest closed set $F \subset X$ with $\mu[X \setminus F] = 0$, and will be denoted $\mathrm{Supp}\,\mu$. On the other hand, when we say that μ is concentrated on $A \subset X$, this only means that $\mu[X \setminus A] = 0$, without A being necessarily closed.

If X is a metric space, we shall equip it with the topology induced by its distance, and denote by $B(x, r)$ the ball of radius r and center x. We shall denote by $\mathrm{Lip}(X)$ the set of all Lipschitz functions on X; we shall also denote by $P_p(X)$ the space of Borel probability measures μ on X with finite moment of order p, in the sense that $\int d(x_0, x)^p\, d\mu(x) < +\infty$ for some (and thus any) $x_0 \in X$.

When X is a Banach space and X^* its topological dual, we shall denote by $\langle \cdot, \cdot \rangle$ the duality bracket between X and X^*. A particular case of this is the scalar product in a Hilbert space.

If φ is a convex function on a Banach space X, then φ^* will stand for its dual convex function, in the sense of Legendre-Fenchel duality. The subdifferential of φ will be denoted by $\partial\varphi$, and identified with its graph, which is a subset of $X \times X^*$. Basic definitions for these objects are recalled

in Chapter 2. From Chapter 3 on, we shall abbreviate "proper lower semicontinuous convex function" into just "convex function".

When X is a smooth Riemannian manifold, or a Banach space, and F is a continuous function on X, we shall denote by DF its differential map, and by $DF(x) \cdot v$ its first-order variation at some point $x \in X$, along some tangent vector v.

When X is a smooth Riemannian manifold, we shall denote by $T_x X$ the tangent space at a point x, and by $\langle \cdot, \cdot \rangle_x$ the scalar product on $T_x X$ defined by the Riemannian structure. We shall denote by $\mathcal{D}(X)$ the space of C^∞ functions on X with compact support, and by $\mathcal{D}'(X)$ the space of distributions on X. We define the gradient operator ∇ on $\mathcal{D}(X)$ by the identity $\langle \nabla F(x), v \rangle_x = DF(x) \cdot v$; so $\nabla F(x)$ belongs to $T_x X$, while $DF(x)$ lies in $(T_x X)^*$. We shall denote by $\nabla \cdot$ the divergence operator, which is the adjoint of ∇ on $\mathcal{D}(X)$. The gradient operator acts on real-valued functions, while the divergence operator acts on vector fields. We also define the Laplace operator Δ by the identity $\Delta F = \nabla \cdot \nabla F$. By duality, all these operations are extended to $\mathcal{D}'(X)$. Of course, if $X = \mathbb{R}^n$, then

$$\nabla F = \left(\frac{\partial F}{\partial x_1}, \ldots, \frac{\partial F}{\partial x_n} \right), \qquad \nabla \cdot u = \sum_{i=1}^n \frac{\partial u_i}{\partial x_i}, \qquad \Delta F = \sum_{i=1}^n \frac{\partial^2 F}{\partial x_i^2}.$$

We also denote by D^2 the Hessian operator on X. Of course, if $X = \mathbb{R}^n$, then $D^2 F(x)$ can be identified with the Hessian matrix $(\partial^2 F(x) / \partial x_i \, \partial x_j)$.

The space of absolutely continuous (with respect to Lebesgue measure) probability measures on $\mathbb{R}^n$ will be denoted by $P_{\mathrm{ac}}(\mathbb{R}^n)$; it can be identified with a subspace of $L^1(\mathbb{R}^n)$. The space of absolutely continuous probability measures with finite moments up to order 2 will be denoted by $P_{\mathrm{ac},2}(\mathbb{R}^n)$.

The Aleksandrov Hessian of a convex function φ on $\mathbb{R}^n$ will be denoted by $D^2_A \varphi$; it is only defined almost everywhere in the interior of the domain of φ. It should not be mistaken for the distributional Hessian of φ, denoted by $D^2_{\mathcal{D}'} \varphi$. The Hessian measure of φ will be denoted $\det_{\mathrm{H}} D^2 \varphi$. All these notions will be explained within the text (see subsections 2.1.3 and 4.1.4). We shall use consistent notations for Laplace operators: the trace of $D^2_A \varphi$ (resp. $D^2_{\mathcal{D}'} \varphi$) will be denoted by $\Delta_A \varphi$ (resp. $\Delta_{\mathcal{D}'} \varphi$).

Whenever Ω is an open subset of $\mathbb{R}^n$ and $k \in \mathbb{N}$, we denote by $C^k(\Omega)$ the space of functions u which are differentiable up to order k; and, whenever $\alpha \in (0, 1)$, we denote by $C^{k,\alpha}(\Omega)$ the space of functions u for which all partial derivatives at order k are Hölder-continuous with exponent α.

Whenever Ω is a smooth subset of $\mathbb{R}^n$, the group of all diffeomorphisms $s : \Omega \to \Omega$ with $\det(\nabla s) \equiv 1$ will be denoted by $G(\Omega)$.

We shall refer to a measurable set $X \subset \mathbb{R}^n$ as a *small set* if it has Hausdorff dimension at most $n - 1$.

The vector space of real $n \times n$ matrices will be denoted by $M_n(\mathbb{R})$. The trace of a matrix M will be denoted by tr M. The $n \times n$ identity matrix will be denoted by I_n. Whenever M is an element of $M_n(\mathbb{R})$, its transposed matrix will be denoted by M^T; thus $M^T = (m'_{ij})$ with $m'_{ij} = m_{ji}$. The sets of symmetric matrices ($M^T = M$), symmetric matrices with nonnegative eigenvalues ($M \geq 0$), antisymmetric matrices ($M^T = -M$) and orthogonal matrices ($MM^T = I_n$) will be respectively denoted by $S_n(\mathbb{R})$, $S_n^+(\mathbb{R})$, $A_n(\mathbb{R})$ and $O_n(\mathbb{R})$.

Finally, let us say a word about where to find the definitions of the basic objects in optimal mass transportation: the notations $I[\pi]$, $\Pi(\mu, \nu)$, $J(\varphi, \psi)$, Φ_c are defined in Theorem 1.3 of Chapter 1; $\mathcal{T}_c(\mu, \nu)$ in formula (5); $W_p(\mu, \nu)$ and $\mathcal{T}_p(\mu, \nu)$ in Theorem 7.3 of Chapter 7.

Contents

Introduction

1. Formulation of the optimal transportation problem

Assume that we are given a pile of sand (say), and a hole that we have to completely fill up with the sand.

Obviously, the pile and the hole must have the same volume. Let us normalize the mass of the pile to 1. We shall model both the pile and the hole by probability measures μ, ν, defined respectively on some measure spaces X and Y. Whenever A and B are measurable subsets of X and Y respectively, $\mu[A]$ gives a measure of how much sand is located inside A; and $\nu[B]$ of how much sand can be piled in B.

Moving the sand around needs some effort, which is modelled by a measurable **cost function** defined on $X \times Y$. Informally, $c(x, y)$ tells how much it costs to transport one unit of mass from location x to location y. It is natural to assume at least that c is measurable and nonnegative. One should not a priori exclude the possibility that c takes infinite values, and so c should be a measurable map from $X \times Y$ to $\mathbb{R}_+ \cup \{+\infty\}$.

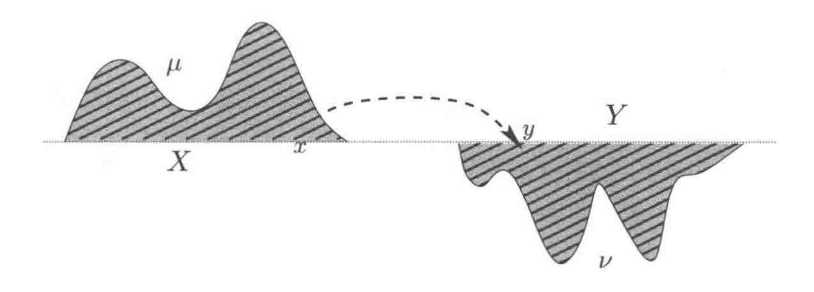


Figure 1. The mass transportation problem

In this book a central question is the following

Basic problem: *How to realize the transportation at minimal cost?*

Before studying this question, we have to make clear what a way of transportation, or a **transference plan**, is. We shall model transference plans by probability measures π on the product space $X \times Y$. Informally, $d\pi(x, y)$ measures the amount of mass transferred from location x to location y. We do not a priori exclude the possibility that some mass located at point x may be split into several parts (several possible destination y's). For a transference plan $\pi \in P(X \times Y)$ to be admissible, it is of course necessary that all the mass taken from point x coincide with $d\mu(x)$, and that all the mass transferred to y coincide with $d\nu(y)$. This means

$$\int_Y d\pi(x,y) = d\mu(x), \qquad \int_X d\pi(x,y) = d\nu(y).$$

More rigorously, we require that

$$\pi[A \times Y] = \mu[A], \qquad \pi[X \times B] = \nu[B], \tag{1}$$

for all measurable subsets A of X and B of Y. This is equivalent to stating that for all functions φ, ψ in a suitable class of test functions,

$$\int_{X\times Y} [\varphi(x) + \psi(y)]\, d\pi(x,y) = \int_X \varphi(x)\, d\mu(x) + \int_Y \psi(y)\, d\nu(y). \tag{2}$$

In general, the natural set of admissible test functions for (φ, ψ) is $L^1(d\mu) \times L^1(d\nu)$, or equivalently $L^\infty(d\mu) \times L^\infty(d\nu)$. In most situations of interest, this class can be narrowed to just $C_b(X) \times C_b(Y)$, or $C_0(X) \times C_0(Y)$; we shall discuss this more precisely later on.

Those probability measures π that satisfy (1) are said to have **marginals** μ and ν, and will be the admissible transference plans. We shall denote the set of all such probability measures by

$$\Pi(\mu,\nu) = \Big\{\pi \in P(X \times Y); \quad \text{(1) holds for all measurable } A,\ B\Big\}. \tag{3}$$

This set is always nonempty, since the tensor product $\mu \otimes \nu$ lies in $\Pi(\mu, \nu)$ (this corresponds to the most stupid transportation plan that one may imagine: any piece of sand, regardless of its location, is distributed over the entire hole, proportionally to the depth).

We now have a clear mathematical definition of our basic problem. In this form, it is known as

Kantorovich's optimal transportation problem:

$$\textit{Minimize} \quad I[\pi] = \int_{X\times Y} c(x,y)\, d\pi(x,y) \qquad \textit{for } \pi \in \Pi(\mu,\nu). \tag{4}$$

This minimization problem was studied in the forties by Kantorovich [**175, 176**] — who was awarded a Nobel prize for related work in economics! That the optimal transference problem is related to basic questions in economics becomes clear if one thinks of μ as a density of production units, and of ν as a density of consumers. For a given transference plan π, the nonnegative (possibly infinite) quantity $I[\pi]$ will be called the **total transportation cost** associated to π. The **optimal transportation cost** between μ and ν is the value

$$\mathcal{T}_c(\mu,\nu) = \inf_{\pi \in \Pi(\mu,\nu)} I[\pi]. \tag{5}$$

The optimal π's, i.e. those such that $I[\pi] = \mathcal{T}_c(\mu,\nu)$, if they exist, will be called **optimal transference plans**.

Readers with probabilistic minds will already have translated the Kantorovich problem into its equivalent

Probabilistic interpretation:

Given two probability measures μ and ν, minimize the expectation

$$I(U,V) = \mathbf{E}\left[c(U,V)\right] \tag{6}$$

over all pairs (U,V) of random variables U in X, and V in Y, such that $\mathrm{law}(U) = \mu$, $\mathrm{law}(V) = \nu$.

For those who are not so familiar with basic probability theory, we recall that a random variable U in X is a measurable map with values in X, defined on a probability space Ω equipped with a probability measure $\mathbf{P}$; that the **law** of U is the probability measure μ on X defined by $\mu[A] = \mathbf{P}[U^{-1}(A)]$; and that the expectation stands for the integral with respect to $\mathbf{P}$. Transference plans $\pi \in \Pi(\mu,\nu)$ are all possible laws of the couple (U,V). Such a π is often said to be the **joint law** of the random variables U and V; one also says that it constitutes a **coupling** of U and V.

As we shall explain in the next section, Kantorovich's problem is a relaxed version of the original mass transportation problem considered by Monge [**209**]. Monge's problem is just the same as Kantorovich's, except for one thing: it is additionally required that *no mass be split.* In other words, to each location x is associated a unique destination y. In terms of random variables, this requirement means that we ask for V to be a function of U in (6). In terms of transference plans, it means that we ask for π in (4) to have the special form

$$d\pi(x,y) = d\pi_T(x,y) \equiv d\mu(x)\,\delta[y = T(x)], \tag{7}$$

where T is a measurable map $X \to Y$. The probability measure appearing on the right-hand side of (7) can also be written $(\mathrm{Id} \times T)\#\mu$; it is the probability measure on $X \times Y$ satisfying the following characteristic property: whenever ζ is a nonnegative (say) measurable function on $X \times Y$, then

$$\int_{X\times Y} \zeta(x,y)\, d\pi_T(x,y) = \int_X \zeta(x, T(x))\, d\mu(x). \tag{8}$$

In particular, the associated total transportation cost is

$$I[\pi_T] = \int_X c(x, T(x))\, d\mu(x).$$

What is the condition that T in (7) should satisfy for π_T to belong to $\Pi(\mu,\nu)$? Well, in view of (8), condition (2) translates into

$$\int_X \big[\varphi(x) + \psi \circ T(x)\big]\, d\mu(x) = \int_X \varphi(x)\, d\mu(x) + \int_Y \psi(y)\, d\nu(y).$$

Cancelling out $\int \varphi\, d\mu$ on both sides, we recover the condition

$$\int_X (\psi \circ T)\, d\mu = \int_Y \psi\, d\nu. \tag{9}$$

This identity should hold true for all nonnegative ψ, or equivalently for all $\psi \in L^1(d\nu)$, or equivalently for all $\psi \in L^\infty(d\nu)$. More precisely: for all $\psi \in L^1(d\nu)$, the measurable function $\psi \circ T$ should lie in $L^1(d\mu)$, and the values of both integrals in (9) should coincide.

Equivalently, in terms of measurable subsets, the condition for π_T to belong to $\Pi(\mu,\nu)$ can be recast as

$$\text{for any measurable set } B \subset Y, \quad \nu[B] = \mu[T^{-1}(B)]. \tag{10}$$

Whenever the equivalent conditions (9) or (10) are satisfied, we shall write

$$\nu = T\#\mu$$

and say that ν is the **push-forward**, or **image measure** of μ by T; or that T **transports** μ onto ν. Of course, the law of a random variable U defined on a probability space Ω equipped with a probability $\mathbf{P}$ is nothing but the image measure $U\#\mathbf{P}$.

We can now formulate a strengthened version of Kantorovich's problem.

Monge's optimal transportation problem:

$$\textit{Minimize} \quad I[T] = \int_X c(x, T(x))\, d\mu(x)$$

over the set of all measurable maps T such that $T\#\mu = \nu$.

This problem has been famous for quite a time. Nearly hundred years after its first formulation by Monge, the Academy of Paris offered a prize for its solution [**111**], and awarded it to Appell [**18**]. In retrospect, the choice of the Academy appeared to be debatable, and neither Appell's manuscript nor that of the other competitors were bringing satisfactory answers: see the account by Ghys in *Images des Mathématiques* [**154**]. It would take another century before the full wealth of the subject start to emerge...

In these notes, we shall use the name "Monge-Kantorovich problem" for either Kantorovich's or Monge's minimization problem.

Example (Dirac mass). Assume that ν is a Dirac mass: $\nu = \delta_a$. Then there is a unique element in $\Pi(\mu,\nu)$ (all the mass should be transported to a), and obviously,

$$\mathcal{T}_c(\mu,\delta_a) = \int_X c(x,a)\,d\mu(x). \tag{11}$$

Example (The discrete case). Suppose X and Y are discrete spaces where all points have the same mass:

$$\mu = \frac{1}{n}\sum_{i=1}^n \delta_{x_i}, \qquad \nu = \frac{1}{n}\sum_{j=1}^n \delta_{y_j}.$$

Any measure in $\Pi(\mu,\nu)$ can clearly be represented as a bistochastic $n\times n$ matrix $\pi = (\pi_{ij})_{i,j}$. Here by bistochasticity we mean that all the π_{ij} are nonnegative and that

$$\forall j,\quad \sum_i \pi_{ij} = 1; \qquad \forall i,\quad \sum_j \pi_{ij} = 1.$$

So in this case the Kantorovich problem reduces to

$$\inf\left\{\frac{1}{n}\sum_{ij}\pi_{ij}\,c(x_i,y_j);\quad \pi\in\mathcal{B}_n\right\},$$

with $\mathcal{B}_n$ denoting the set of bistochastic $n\times n$ matrices.

This is a linear minimization problem on the bounded convex set $\mathcal{B}_n \subset M_n(\mathbb{R})$. By **Choquet's minimization theorem**, we know that this problem admits solutions which are **extremal points** of $\mathcal{B}_n$, i.e. elements of $\mathcal{B}_n$ which cannot be written as a nontrivial convex combination of two points in $\mathcal{B}_n$. By **Birkhoff's theorem**, these extremal points are the *permutation matrices*, i.e. those matrices such that $\pi_{ij} = \delta_{j,\sigma(i)}$ for some permutation σ

of $\{1,\ldots,n\}$ (here δ_{jk} is the Kronecker symbol: $\delta_{jk}=1$ if $j=k$, 0 otherwise). Thus, optimal transference plans in Kantorovich's problem coincide with solutions of Monge's problem

$$\inf\left\{\frac{1}{n}\sum_i c(x_i,y_{\sigma(i)});\quad \sigma\in\mathcal{S}_n\right\}$$

with $\mathcal{S}_n$ standing for the set of permutations in $\{1,\ldots,n\}$. We also see that the optimal transportation problem corresponds to finding an **optimal matching** between the source points x_i and the target points y_j.

Remark. This line of reasoning fails in the continuous setting: in general, if μ and ν are absolutely continuous with respect to Lebesgue measure, there exist extremal points in $\Pi(\mu,\nu)$ which are not concentrated on any graph.

2. Basic questions

Once the above preliminaries are set, the most natural basic question is

Question 1: *Do there exist minimizers to the Monge-Kantorovich problem, and can one characterize them?*

And once a (complete or partial) answer has been given, it is natural to examine

Question 2: *What information on μ, ν does the knowledge of the optimal transportation cost $\mathcal{T}_c(\mu,\nu)$ bring?*

The answers to both questions depend heavily on the structure of the space, and on the cost function. The answer to Question 1 also depends on the "regularity" of μ,ν. Let us illustrate this on some examples, most of which will be examined again later on. It will be sufficient to consider the situation when $X=Y=\mathbb{R}^n$, $c(x,y)=|x-y|^p$, $0<p<+\infty$, and μ,ν are compactly supported.

• For $p>1$, the *strict convexity* of c guarantees that, if μ,ν are absolutely continuous with respect to Lebesgue measure, then there is a unique solution to the Kantorovich problem, which turns out to be also the solution to the Monge problem. The same result holds true under the weaker assumption that μ does not give positive mass to any set with finite $(n-1)$-dimensional Hausdorff measure.

• One can then give a geometrical characterization of the optimal maps T. For instance, in the case of a *quadratic cost*, $p=2$, and in this case only (for $n>1$), these optimal maps are the (restrictions of) gradients of convex functions on $\mathbb{R}^n$. In particular, they are monotone and orientation-preserving.

• On the other hand, it is easy to construct measures μ, ν in $\mathbb{R}^n$, which charge a "small" set (a set of Hausdorff dimension at most $n-1$ in $\mathbb{R}^n$, for instance a line segment in dimension 2), and such that optimal transference plans in Kantorovich's problem have to split mass, so that solutions to the Kantorovich and Monge problems do *not* coincide. Worse, there will be no solution to the Monge problem!

• But if $p=1$, even the assumption that μ and ν do not give positive mass to small sets will not ensure that the solutions to the Monge and Kantorovich problems coincide (one can construct counter-examples concentrated on a set of dimension $n-\varepsilon$, ε as small as desired). If μ and ν are absolutely continuous with respect to the Lebesgue measure, then there are solutions of the Monge problem which are also solutions of the Kantorovich problem; however it is easy to check that there is no uniqueness.

• For $p<1$, the situation is again different: there is in general no solution of the Monge problem, except if μ and ν are concentrated on disjoint sets. And when they exist, these solutions are locally orientation-reversing, which is the opposite of what happens for $p>1$.

These various examples were mainly extracted from [**15, 151, 205**]. Another illustration of the different geometrical properties of the cases $p=1$ and $p=2$ is provided by Problem 1 in Chapter 10.

As for Question 2, it is less sensitive to the cost. But it does depend on it, as shown by the following examples.

• Consider, in a complete separable metric space (X,d), the cost $c(x,y)=d(x,y)^p$, for some given $p\in(0,\infty)$. Then

$$\mathcal{T}_c^{\min(1,1/p)} = \left\{ \inf_{\pi\in\Pi(\mu,\nu)} \int_{X\times X} d(x,y)^p \, d\pi(x,y) \right\}^{\min(1,1/p)} \tag{12}$$

is a *metric* on the space $P(X)$, which metrizes *weak convergence* of probability measures, as soon as one controls the moments of order p. Here is a more precise statement: let $(\mu_k)_{k\in\mathbb{N}}$ be a sequence of probability measures such that for some $x_0\in X$, the sequence of measures $d\mu_k(x)\,d(x,x_0)^p$ is tight; then μ_k converges to μ weakly if and only if $\mathcal{T}_c(\mu_k,\mu)\longrightarrow 0$. We recall that a family ρ_k of nonnegative measures on a topological space X is said to be tight if for all $\varepsilon>0$ there is a compact set K_ε with $\sup_k \rho_k[X\setminus K_\varepsilon]\le\varepsilon$.

• On the other hand, consider the special cost $c(x,y)=1_{x\neq y}$ (which is a peculiar distance!). Then the total transportation cost is a familiar object:

$$\mathcal{T}_c(\mu,\nu) = \frac{1}{2}\|\mu-\nu\|_{TV}, \tag{13}$$

with the subscript TV standing for "total variation". This identity is a particular case of **Strassen's theorem**; it is often stated in the probabilistic version

$$\text{(14)} \qquad \inf \mathbf{E}\,[X \neq Y] = \sup \Big\{ \mu[F] - \nu[F]; \quad F \text{ closed} \Big\}.$$

Of course, it implies that $\mathcal{T}_c$ metrizes the *strong* topology on the space of measures.

Remark. Identity (13) is intuitively clear: for the cost function $1_{x \neq y}$, the optimal transportation cost is obtained when all the mass that can stay in place (all the mass shared by μ and ν) does indeed stay in place. If one sends all the remaining mass of μ onto the remaining mass of ν, the corresponding total cost is (with sloppy notation)

$$\int_{\mu-\nu \geq 0} d(\mu - \nu) = \frac{1}{2} \left[\int_{\mu-\nu \geq 0} d(\mu - \nu) + \int_{\mu-\nu \leq 0} d(\nu - \mu) \right]$$

since $\int_{\mu-\nu=0} d(\mu - \nu) = 0$, and $\int_X d(\mu - \nu) = 0$.

In Chapter 1, we shall see that identity (13) (as well as the more general version of Strassen's theorem) can be seen as a consequence of some basic results in the theory of optimal transportation.

Although the ordering of our two basic questions seems quite natural (at least to the author of these notes!), it turns out that Question 2 was reasonably well mastered long before Question 1. Even back in the seventies and earlier, many researchers in probability theory understood the great interest that the mass transportation problem could have for them, as a tool of measuring distances between probability measures in more or less complicated settings.

The history of Question 1 is much more agitated. Ironically, Monge's original problem was extremely difficult, for two reasons. First, as a general rule the Monge problem is much more tricky than the Kantorovich problem. Secondly, the particular cost function which he considered, namely the distance function $c(x, y) = |x - y|$, is very degenerate from the point of view of convexity properties. In spite of Sudakov's important work [**242**] at the end of the seventies, it is certainly fair to state that it took two more decades before the Monge problem could begin to be reasonably well understood. In fact, Sudakov "proved" the existence of a minimizer to the Monge problem with a distance cost function, but his argument was not completely correct, as pointed out by Alberti, Kirchheim and Preiss (see the explanations in Ambrosio and Pratelli [**15**, sect. 1]; see also Section 8 in the same reference). Even though Sudakov's proof was carefully fixed in [**12, 15**], it is striking to realize that the first published correct (so it seems!) proof of existence for this cost function was the difficult 1999 memoir by Evans and

Gangbo [**135**], which partly relied on the theory of p-Laplace equations, and in which somewhat strong regularity assumptions were imposed on the marginals μ and ν...

Even if we forget about this degeneracy of the distance cost function, the following simple remark should be sufficient to understand why Monge's problem is in general so tricky. Assume that the probability measures μ and ν are defined on $\mathbb{R}^n$ (or a smooth manifold) and are absolutely continuous with respect to Lebesgue measure:

$$d\mu(x) = f(x)\,dx, \qquad d\nu(y) = g(y)\,dy.$$

Make the ansatz (in general unrealistic) that the T which we are looking for is a C^1 diffeomorphism (from $\mathbb{R}^n$ to $\mathbb{R}^n$, or more generally between smooth manifolds). Then, the reader can check, as an exercise, that the **change of variable formula** enables one to rewrite formula (10) as

$$f(x) = g(T(x))\,|\det \nabla T(x)|. \tag{15}$$

By usual standards this is a "highly nonlinear" constraint on T, very difficult to handle by the classical tools of calculus of variations. This is quite in contrast with the constraint (1) for the Kantorovich problem, which is linear.

In the terminology of the calculus of variation, Kantorovich's problem can be seen as a **relaxed** version of Monge's problem. Relaxation here means that, starting from the Monge problem, one extends the class of objects on which the infimum is taken, by embedding the space of measurable functions $T : X \to Y$ into the space $L^0(X; P(Y))$ of measurable functions with values in $P(Y)$. Explicitly, this embedding is defined by

$$\mathcal{I} : T \longmapsto [x \mapsto \delta_{T(x)}], \tag{16}$$

with $\delta_{T(x)}$ standing for the Dirac mass at $T(x) \in Y$. Under this procedure, the nonlinear constraint on T formally turns into a linear constraint on $\mathcal{I}(T)$; one can use this fact to define a minimization problem on $L^0(X; P(Y))$ with a linear constraint, and this new problem turns out to be Kantorovich's problem.

The very same idea has been popularized in the calculus of variations, and the embedding (16) is the starting point of the theory of **Young measures** [**270, 271, 272**]. We refer to [**15, 149**] for a user-friendly comparison between the ideas of Kantorovich and those of Young. Using approximation arguments inspired by the theory of Young measures, Gangbo [**149**, Appendix A] and Ambrosio [**12**, Theorem 2.1] proved under quite general assumptions (continuous cost function) that the *values* of the infima in both the Monge and the Kantorovich problems have to coincide, if μ has no atom, i.e. there is no $x \in X$ with $\mu[\{x\}] > 0$. Their proof is based on the fact that the range of $\mathcal{I}$ is dense in $L^1(X; P(Y))$, equipped with the measure μ.

Probably because of the evolution of mathematical feelings, nowadays it is Kantorovich's problem which appears most natural (at least to the author), which is why we prefer to present Monge's problem as a particular case of Kantorovich's, rather than to introduce Kantorovich's problem as a relaxed version of Monge's.

3. Overview of the course

In Chapter 1, we begin our study by a presentation of the Kantorovich duality, which is a very powerful tool, from both the theoretical and the numerical points of view. It was Kantorovich who formally pointed out this dual formulation (see Theorem 1.3), and, together with Rubinstein, proved it rigorously for a compact space equipped with a distance cost function, in which case it takes a simpler form (see Theorem 1.14). The Kantorovich-Rubinstein theorem was rewritten in the seventies by Dudley [**124**, Lect. 20], motivated by problems of mathematical statistics (see [**127**, Chapter 11] for a more recent exposition), and later pushed to a high degree of abstraction by several authors [**225**]. Our setting here will be very general (Polish space, arbitrary lower semi-continuous cost function).

Then, in Chapter 2, we review the most important results about the existence and characterization of the optimal transference plans, i.e. Question 1 above. As we shall see, it is usually very simple to give a proof of existence of a minimizer for the Kantorovich problem, but it is much trickier to do the same for the Monge problem. From the end of the eigthies on, a systematic and elegant theory for the Monge problem was developed by (in alphabetic order) Brenier, Evans, Gangbo, Knott and Smith, McCann, Rachev, Rüschendorf, and others, with particular emphasis on the quadratic cost $c(x, y) = |x - y|^2$ in $\mathbb{R}^n$. The basic result for this quadratic cost (Theorem 2.12) was discovered at least twice, first by Knott and Smith, then by Brenier, who was motivated by considerations arising from fluid mechanics. Their results were generalized by McCann on one hand, Rachev and Rüschendorf on the other. For many researchers, however, it is the name of Brenier which is associated with this theorem, because he was one of the first to unveil the potential applications of mass transportation to problems of classical mechanics or mathematical physics.

Apart from the quadratic setting, similar results concern the cost functions $|x - y|^p$, or more generally $d(x, y)^p$ on a Riemannian manifold. In this course, we mainly focus on the quadratic cost function, then just review the basic known results for other costs. As we already said, we recommend the notes by Evans [**134**] and especially those by Ambrosio [**12**], and by Ambrosio and Pratelli [**15**] for a much more complete treatment of the cost function $c(x, y) = |x - y|$.

In Chapter 3, we develop a little bit on the motivations which led Brenier to study the Monge-Kantorovich problem, and explain Brenier's polar factorization theorem (Theorem 3.8), which is essentially equivalent to the optimal transportation theorem. This chapter contains an overview of Brenier's original motivations in fluid mechanics, but does not dig deeply into the subject, which probably would deserve a book on its own.

From the point of view of partial differential equations, the Knott-Smith-Brenier theorem implies existence and uniqueness of the solution, in a very weak sense, to a certain Monge-Ampère equation. For this equation a difficult regularity theory was developed by Caffarelli and by Urbas, using quite different tools (see Theorem 4.14). Their work allows one to study some smoothness properties of the optimal transference plan, up to now only in the quadratic case. We quickly review the theory in Chapter 4. This chapter is the only one in which the proofs of the most important results are not provided. Indeed, developing the arguments in this chapter would have meant essentially writing a second book; moreover, the excellent set of notes by Urbas [**255**] and the reference book by Gutiérrez [**164**] can be consulted by readers who want to know more about regularity.

Next, in Chapter 5, we introduce two simple but fundamental notions due to McCann: **displacement interpolation** and **displacement convexity**. For many applications, displacement convexity will turn out to be the right notion of convexity. We shall spend some time explaining various ways to look at it, and some of its applications.

Chapter 6 penetrates into a rather different world, which is the universe of geometric inequalities. The prototype of such inequalities is certainly the isoperimetric inequality. We shall explain in this chapter how mass transportation provides amazingly powerful tools to study some functional inequalities with geometric content. This chapter follows a series of works initiated by McCann and continued by Alesker, Barthe, Cordero-Erausquin, Dar, Milman, Schmuckenschläger, and others. In the end, we also present a new proof of the optimal Sobolev inequality, devised jointly by Cordero-Erausquin, Nazaret and the author.

On the whole, in Chapters 2 to 6, we will only be interested in the minimizers for the Monge-Kantorovich problem: characterize them, use their existence for various applications, etc. To go further, we shall need to be interested in Question 2 above. This is why in Chapter 7 we review the properties of the **Monge-Kantorovich distances** or **Wasserstein distances** (see Theorem 7.3), which are distances induced by the value of the optimal transportation cost when the cost is a power of a distance. This question is actually much simpler than Question 1, and its answer has been

known for some time (see Theorem 7.12). Many many names are associated with these distances (Dall'Aglio, Fréchet, Gini, Hoeffding, Hutchinson, Kantorovich, Rubinstein, Tanaka, Wasserstein, and others) and we make no attempt to review the existing literature on the subject. The properties of the Wasserstein distances depend very little on the underlying geometrical structure, and the results in this chapter hold in extreme generality (Polish spaces).

A most important particular case is the Wasserstein distance of order 2, or (by abuse of terminology) the **quadratic Wasserstein distance**,

$$W_2(\mu,\nu) = \sqrt{\mathcal{T}_{d^2}(\mu,\nu)}. \tag{17}$$

This distance plays a central role among Monge-Kantorovich distances, just as L^2 plays a central role in the family of L^p spaces. An application to the study of the Boltzmann equation, due to Tanaka, is expounded (Theorem 7.23); even though it is by now somewhat outdated, it is interesting and not well-known.

Chapter 8 is a key chapter: there we introduce and study a differential, dynamical formulation of the mass transportation problem. This idea, which leads to a reformulation from a fluid mechanics point of view, goes back to Benamou and Brenier (see Theorem 8.1). It was exploited by Otto at the end of the nineties to develop an extremely influential geometric view about optimal transportation.

As an illustration of the power of this differential point of view, in Chapter 9, we explain how it enables one to link mass transportation with several classes of functional inequalities which are useful in many different contexts: logarithmic Sobolev inequalities, entropy-entropy production inequalities, transportation inequalities. These links were first studied systematically by Otto and the author (see the formal Theorem 9.2).

Not all chapters are of equal status. Chapters 1, 2, 4 and 7 contain the presentation of the basic theory. These are certainly the chapters which should first be read by a graduate student — particularly Chapter 2. All the proofs of the main results in these chapters are given in detail, except for the last part of Chapter 4. Chapter 5 is crucial for applications, and also deserves careful reading by students; the proofs there are rather easy, but the conceptual gain may be very rewarding. Chapter 3 is not difficult, but part of it deals with more original subjects, related to hydrodynamic equations. A reader who would like to know more about the basics of the huge theory of these equations may consult an introductory book on the subject, like Chorin and Marsden [**95**], intended for mathematicians. The last part of Chapter 4 touches on a very, very tricky subject, namely the regularity theory for fully nonlinear elliptic equations; but, precisely for

this reason, the subject is reduced to just a review of the results, and this chapter should not be so difficult to go through. Finally, Chapters 6, 8 and 9 are more advanced; especially the last two chapters presuppose some basic notions in partial differential equations and functional analysis, but not more than can be found in any elementary textbook on the subject, like the excellent reference [**133**], which we shall quote many times.

As a general prerequisite for the course, the reader will be assumed to have some basic background knowledge in analysis, especially measure theory, and a little bit of functional analysis. The necessary material can be found in classical pedagogical references such as Rudin [**231**], Lieb and Loss [**190**] or Brézis [**68**]; but many other sources would do. On some rare occasions the reader may wish to consult slightly more advanced textbooks such as Rudin [**232**] (for functional analysis), Evans and Gariepy [**136**] (for fine regularity properties of convex functions, in particular), Billingsley [**43**] or Dudley [**127**] (for probability theory in abstract Polish spaces); but these sources should definitely not be needed in a first reading. If the reader is not familiar with the notion of Hausdorff dimension, we recommend that he/she just forget about it and skip those statements which involve this concept, since they are not of primary importance here. For instance, if a theorem assumes that some measure does not give mass to small sets, just make the stronger assumption that it is absolutely continuous with respect to the Lebesgue measure.

There are exercises disseminated all over the text, most of them easy ones whose primary goal is to help the reader's understanding by making him/her manipulate the basic concepts a little bit. Some exercises however are more tricky. Chapter 10 gathers longer problems, most of which are taken from research papers. We chose to gather all these problems together for two reasons: first, because this enables to introduce a great variety of illustrations and applications without being led to an excess of digressions within the main text; and secondly, because the solution of one of these problems often requires material which appears at various places in the book. These problems should therefore be of interest for readers who wish to have a synthetic view of the topic.

Apart from the index, a "Table of Short Statements" has been added in the back of the book; it contains short statements for all the theorems, propositions, definitions, corollaries, etc. which are proven or explained in the main text. Glancing through this table might be a quick way to locate some particular theorem in the book.

Had the author had more time and/or courage, he would have been really pleased to add one or several chapters on original models derived from

optimal transportation; notable examples range from the theory of semi-geostrophic equations to electromagnetism. The semi-geostrophic equations provide a beautiful example of a Hamiltonian system arising in fluid mechanics, in which the quadratic Wasserstein distance, amazingly, plays an explicit role. They were first studied by Cullen and collaborators [**109, 108, 105**], then by various mathematicians [**38, 106, 107**]. It is absolutely remarkable that Cullen's group was led to the intuition of Brenier's theorem on the sole basis of physical considerations like stability of some weather patterns. To get an idea of this field, the reader may consult Problem 9 in Chapter 10. As for electromagnetism, there is a series of works by Brenier [**65**] about the interpolation of currents and a suggestion to use the Monge-Ampère equation as a replacement for the Poisson equation arising from Coulomb interaction. Other remarkable applications of optimal transport occur in image processing, meteorology, economy, etc.

Finally, numerical simulations are not discussed in this book, although this topic is of fundamental interest.

Warm-up exercises

The following exercises are specifically intended for students who would not feel so comfortable with some of the basic notions used in this introduction, and would like to practise just a little bit before more serious matters begin. In case the reader needs to review measure theory, he or she can consult for instance Rudin [**231**].

1. About the definition of image measure. Let X be a probability space, equipped with a probability measure μ, and let Y be an abstract set. Given any map $T : X \to Y$, show that Y can be equipped with a structure of measure space, in such a way that T is measurable. Show that the formula $\nu[B] = \mu[T^{-1}(B)]$ uniquely defines a probability measure ν on Y. Prove that this measure is characterized by equation (9) holding true for all bounded measurable ψ. Show that this identity actually holds true for all $\psi \in L^1(d\nu)$.

2. About the change of variable formula. Construct examples in which $T\#\mu = \nu$, $d\mu(x) = f(x)\,dx$, $d\nu(y) = g(y)\,dy$, but formula (15) does not apply, because (a) T is not smooth enough, or (b) T is not one-to-one.

3. About Choquet's minimization theorem. Let K be a compact convex set of a Banach space E, and let $\mathcal{E}(K)$ stand for the set of all extremal points of K. Let $\ell : K \to \mathbb{R}$ be the restriction of a continuous linear functional on E. Recall why ℓ admits one or several minimizers in K. Show that

at least one of these minimizers belongs to $\mathcal{E}(K)$, using **Choquet's representation theorem**, according to which every $x \in K$ is the barycenter of a probability on $\mathcal{E}(K)$:

$$\forall x \in K, \quad \exists \rho_x \in P(\mathcal{E}(K)); \quad x = \int_{\mathcal{E}(K)} y \, d\rho_x(y).$$

Also give an alternative, elementary proof when E is finite-dimensional.

Note: Choquet's representation theorem is related to the **Krein-Milman theorem** , according to which, if K is a compact convex set, then K is the smallest closed convex set containing $\mathcal{E}(K)$. It is however not obvious to go from one statement to the other, because $\mathcal{E}(K)$ might not be closed, even in finite dimension!

4. About Birkhoff's theorem. Let $\mathcal{B}_n$ be the set of all bistochastic $n \times n$ matrices (recall the definition on p. 5). Note that $\mathcal{B}_n$ is convex and compact. The goal is to show that $\mathcal{B}_n$ admits exactly $n!$ extremal points, which are the $n \times n$ permutation matrices.

(i) Show that each permutation matrix is an extremal point.

(ii) Show that we just need to prove the following: whenever M is an extremal point of $\mathcal{B}_n$, all entries of M are either 0 or 1.

(iii) Let $M \in \mathcal{B}_n$, and assume that there is at least one entry of M, say $a_{i_0 j_0}$, which lies in $(0,1)$. Construct a family $a_{i_0 j_1}$, $a_{i_1 j_1}$, $a_{i_1 j_2}$, $a_{i_2 j_2}$, etc. of distinct entries which all lie strictly between 0 and 1, and repeat the process until $j_\kappa = j_0$, or $i_\kappa = i_0$. If the latter event arises first, then destroy $a_{i_0 j_0}$ from the list of chosen entries, so than you only keep a list with an even number of entries. Show that one can modify M by perturbing these entries, and only them, and still have a bistochastic matrix; show finally that M can be written as the midpoint between two distinct elements in $\mathcal{B}_n$, and therefore is not an extremal point of $\mathcal{B}_n$.

Chapter 1

The Kantorovich Duality

In this first chapter, we shall investigate a powerful duality formula due to Kantorovich. In order to emphasize the great generality of this principle, which does not require any underlying geometrical structure, we shall consider here abstract Polish spaces (complete, separable metric spaces). Polish spaces are commonly used in probability theory, see for instance Billingsley [**43**] or Dudley [**127**]. In fact, an even more general setting would be possible here; some of the results below are proven in [**127**] without assuming completeness, and it is shown in [**225**] that the topological setting can sometimes be dispended with (while measurability is a crucial assumption, see [**125**]). On the contrary, most of the rest of these notes, with the important exception of Chapter 7, will take place in a much more restricted framework (Banach space, Euclidean space or Riemannian manifold).

This chapter will also be an opportunity to recall some basic notions about probability measures and topology.

1.1. General duality

1.1.1. Definitions and preliminaries. We start by recalling the basic notions of optimal mass transportation. Let (X, μ) and (Y, ν) be probability spaces, and let c be a nonnegative measurable function on $X \times Y$. Kantorovich's mass transportation problem consists in minimizing the linear functional

$$\pi \longmapsto \int_{X \times Y} c(x, y)\, d\pi(x, y)$$

on the nonempty, convex set $\Pi(\mu, \nu)$, defined as the set of all probability measures on $X \times Y$ with marginals μ on X and ν on Y. More explicitly, $\pi \in \Pi(\mu, \nu)$ if and only if π is a nonnegative measure satisfying

$$\pi[A \times Y] = \mu[A], \qquad \pi[X \times B] = \nu[B], \tag{1.1}$$

for all measurable subsets A of X and B of Y. Note that this definition forces π to be a probability measure. Equivalently, $\pi \in \Pi(\mu, \nu)$ if and only if it is a nonnegative measure on $X \times Y$ such that, for all measurable functions $(\varphi, \psi) \in L^1(d\mu) \times L^1(d\nu)$, or equivalently $L^\infty(d\mu) \times L^\infty(d\nu)$,

$$\int_{X\times Y} [\varphi(x) + \psi(y)]\, d\pi(x, y) = \int_X \varphi\, d\mu + \int_Y \psi\, d\nu. \tag{1.2}$$

It is often convenient to use a narrower class of test functions in (1.2), but this can be done only under some topological assumptions on the measure spaces (X, μ) and (Y, ν). Recall that a **Borel** probability measure is a probability measure defined on the Borel σ-algebra of some topological space, i.e. the σ-algebra generated by open sets. In the sequel, we shall only consider such probability measures, and $P(X)$ will stand for the set of Borel probability measures on X. When X and Y are **Polish** spaces (i.e. complete separable metric spaces), and μ, ν are Borel probability measures, it is sufficient to impose (1.2) for $(\varphi, \psi) \in C_b(X) \times C_b(Y)$ only. When in addition X and Y are **locally compact**, i.e. each point admits a compact neighborhood, then one can even be content with imposing (1.2) for $(\varphi, \psi) \in C_0(X) \times C_0(Y)$. Recall that $C_b(X)$ denotes the space of bounded continuous functions on X, and $C_0(X)$ the space of continuous functions going to 0 at infinity, i.e. those continuous functions φ such that for any $\varepsilon > 0$ there is a compact set $K_\varepsilon \subset X$ satisfying $\sup_{x \notin K_\varepsilon} |\varphi(x)| \leq \varepsilon$; note that $C_0(X) \subset C_b(X)$. This possibility to restrict the class of test functions to the narrower space C_0 when X and Y are locally compact is due to **Riesz' theorem** [**231**, p. 40], which identifies the space $M(X)$ of Borel measures having finite total variation on X with the topological dual of $C_0(X)$.

The basic examples one should keep in mind to appreciate the interest of these definitions are the following. Of course $\mathbb{R}^n$ is Polish and locally compact, and so is any finite-dimensional complete Riemannian manifold. On the other hand, $C_b([0,1];\mathbb{R}) = C([0,1];\mathbb{R})$, endowed with the supremum distance, and $P(\mathbb{R}^n)$, endowed with the weak topology of measures, are Polish spaces, but not locally compact. Both $C([0,1];\mathbb{R})$ and $P(\mathbb{R}^n)$ are typical examples of spaces in which one is led to use mass transportation when studying problems of statistical mechanics.

Exercise 1.1. Let K be a compact of $C([0,1];\mathbb{R})$; show that K has empty interior (of course this a particular case of a well-known theorem: the unit ball of a normed vector space X is not compact if X has infinite dimension;

but here this can be checked more directly). Deduce that $C_0(C([0,1];\mathbb{R})) = \{0\}$, and in particular the conclusion of Riesz' theorem does not apply to $C([0,1];\mathbb{R})$.

Hints: Besides being uniformly bounded, K should satisfy the uniform equicontinuity criterion of **Ascoli's theorem**:

$$\forall \eta > 0 \quad \exists \delta > 0; \qquad |x-y| \leq \delta \Longrightarrow \sup_{f \in K} |f(x) - f(y)| \leq \eta.$$

Using this, for any $u \in K$ it is possible to construct $v \in C([0,1];\mathbb{R}) \setminus K$ with $\|u - v\|_{L^\infty}$ as small as desired.

Exercise 1.2. Show that $L^\infty((0,1))$, $C_b(\mathbb{R}^n)$ are not separable, but that $C_0(\mathbb{R}^n)$ is.

Hint: In the case of both $L^\infty((0,1))$ and $C_b(\mathbb{R}^n)$ one can construct a non-countable family of elements, any two of which are separated by a distance exactly 1.

1.1.2. Duality. It is well-known, and widely used, that a linear minimization problem with convex constraints, like (4), admits a **dual formulation**. In the context of optimal mass transportation, it was introduced by Kantorovich in 1942. He was concerned with the particular case when the cost function is a *distance*: $c(x,y) = d(x,y)$, but in fact his duality theorem holds in considerable generality, as can be seen from Theorem 1.3 below. As mentioned above, there are also versions of this theorem holding in more exotic topological spaces, or even non-topological settings (see [**225**]), but we shall not enter into these tricky variants.

Theorem 1.3 (Kantorovich duality). *Let X and Y be Polish spaces, let $\mu \in P(X)$ and $\nu \in P(Y)$, and let $c : X \times Y \to \mathbb{R}_+ \cup \{+\infty\}$ be a lower semi-continuous cost function.*

Whenever $\pi \in P(X \times Y)$ and $(\varphi, \psi) \in L^1(d\mu) \times L^1(d\nu)$, define

$$I[\pi] = \int_{X \times Y} c(x,y)\, d\pi(x,y), \qquad J(\varphi,\psi) = \int_X \varphi\, d\mu + \int_Y \psi\, d\nu.$$

Define $\Pi(\mu,\nu)$ to be the set of all Borel probability measures π on $X \times Y$ such that for all measurable subsets $A \subset X$ and $B \subset Y$,

$$\pi[A \times Y] = \mu[A], \qquad \pi[X \times B] = \nu[B],$$

and define Φ_c to be the set of all measurable functions $(\varphi,\psi) \in L^1(d\mu) \times L^1(d\nu)$ satisfying

$$\varphi(x) + \psi(y) \leq c(x,y) \tag{1.3}$$

for $d\mu$-almost all $x \in X$, $d\nu$-almost all $y \in Y$ (that is to say, for all (x,y) outside of a $(\mu \otimes \nu)$-negligible set).

Then

$$\inf_{\Pi(\mu,\nu)} I[\pi] = \sup_{\Phi_c} J(\varphi, \psi). \tag{1.4}$$

Moreover, the infimum in the left-hand side of (1.4) *is attained. Furthermore, it does not change the value of the supremum in the right-hand side of* (1.4) *if one restricts the definition of* Φ_c *to those functions* (φ, ψ) *which are bounded and continuous.*

We recall that a function F on a metric space X is said to be lower semi-continuous if it satisfies the characteristic property that, for any $x \in X$,

$$F(x) \leq \liminf_{y \to x} F(y).$$

Let us make a few comments before starting the proof of Theorem 1.3. Further remarks will be made after the proof.

Remarks 1.4. (i) It is not a priori clear that the value of $\sup J$ does not change upon restricting the definition of Φ_c to continuous functions, since it is not so clear that pairs of L^1 functions satisfying (1.3) can be approximated by pairs of continuous functions satisfying the same inequality. However Remark 1.6 below will make this claim plausible. When we wish to emphasize the distinction between these definitions, we shall write informally $\Phi_c \cap C_b$, $\Phi_c \cap L^1$.

(ii) Actually, it is the infimum problem in (1.4) which should be called the dual problem (what will be used in the proof, is that $M(X \times Y)$ is the dual space to $C(X \times Y)$ when X, Y are compact!).

(iii) Theorem 1.3 does not even assume that the value of the infimum is finite.

(iv) As we shall see in the next section, in the case of a distance cost function, there is more to say about this duality.

(v) In this chapter, we do not care whether the supremum of J is achieved or not. This question will be addressed in Chapter 2, see in particular Exercise 2.36.

1.1.3. The shipper's problem. Here is an informal interpretation of Theorem 1.3, which we learnt from Caffarelli. Suppose for instance that you are both a mathematician and an industrialist, and want to transfer a huge amount of coal from your mines to your factories. You can hire trucks to do this transportation problem, but you have to pay them $c(x, y)$ for each ton of coal which is transported from place x to place y. Both the amount of coal which you can extract from each mine, and the amount which each factory should receive, are fixed. As you are trying to solve the associated

Monge-Kantorovich problem in order to minimize the price you have to pay, another mathematician comes to you and tells you "My friend, let me handle this for you: I will ship all your coal with my own trucks and you won't have to worry about what goes where. I will just set a price $\varphi(x)$ for loading one ton of coal at place x, and a price $\psi(y)$ for unloading it at destination y. I will set the prices in such a way that your financial interest will be to let me handle *all* your transportation! Indeed, you can check very easily that for any x and y, the sum $\varphi(x)+\psi(y)$ will always be less that the cost $c(x, y)$ (in order to achieve this goal, I am even ready to give financial compensations for some places, in the form of negative prices!)".

Of course you accept the deal. Now, what Kantorovich's duality tells you is that if this shipper is clever enough, then he can arrange the prices in such a way that you will pay him (almost) as much as you would have been ready to spend by the other method.

1.1.4. Preliminary observation. We shall first prove a subpart of Theorem 1.3, which is completely elementary and should at least make the Kantorovich duality appear less surprising.

Proposition 1.5 (Easy part of the Kantorovich duality). *Under the same assumptions as in Theorem* 1.3,

$$\sup_{\Phi_c \cap C_b} J(\varphi, \psi) \leq \sup_{\Phi_c \cap L^1} J(\varphi, \psi) \leq \inf_{\Pi(\mu,\nu)} I[\pi]. \tag{1.5}$$

Proof. The inequality on the left of (1.5) is trivial, since $C_b(X) \times C_b(Y) \subset L^1(d\mu) \times L^1(d\nu)$. So we only care about the inequality on the right. Let (φ, ψ) in $\Phi_c \cap L^1$, and let π be any element of $\Pi(\mu, \nu)$. Then, by definition of Π,

$$J(\varphi, \psi) = \int_X \varphi \, d\mu + \int_Y \psi \, d\nu = \int_{X\times Y} [\varphi(x) + \psi(y)] \, d\pi(x, y).$$

But inequality (1.3) holds $d\pi(x, y)$-almost everywhere. Indeed, let N_x, N_y be measurable sets such that $\mu[N_x] = 0$, $\nu[N_y] = 0$, and inequality (1.3) holds for $(x, y) \in N_x^c \times N_y^c$. Since π has marginals μ and ν, we can write $\pi[N_x \times Y] = \mu[N_x] = 0$, $\pi[X \times N_y] = \nu[N_y] = 0$, and hence $\pi[(N_x^c \times N_y^c)^c] = 0$. As a consequence,

$$\int_{X\times Y} [\varphi(x) + \psi(y)] \, d\pi(x, y) \leq \int_{X\times Y} c(x, y) \, d\pi(x, y) = I[\pi]. \tag{1.6}$$

The inequality (1.5) follows from (1.6) upon taking the supremum on the left-hand side, and the infimum on the right-hand side. □

Remark 1.6. It follows from Proposition 1.5 that the duality $\sup_{\Phi_c \cap C_b} J = \inf I$ automatically implies $\sup_{\Phi_c \cap L^1} J = \sup_{\Phi_c \cap C_b} J$.

1.1.5. A formal proof. Let us now give a formal proof of Theorem 1.3. The idea, which is standard in problems of this kind, is to rewrite the constrained infimum problem as an inf sup problem, and exchange the two operations by formally applying a **minimax principle**, i.e. replacing an "inf sup" by a "sup inf". For readers who are not familiar with this method, it is certainly more important to understand the formal proof, than the rigorous one.

So let us write

$$\inf_{\pi\in\Pi(\mu,\nu)} I[\pi] = \inf_{\pi\in M_+(X\times Y)} \left(I[\pi] + \begin{cases} 0 \text{ if } \pi\in\Pi(\mu,\nu) \\ +\infty \text{ else} \end{cases} \right). \tag{1.7}$$

Here $M_+(X\times Y)$ denotes the space of nonnegative Borel measures on $X\times Y$. The function appearing inside brackets in the right-hand side of (1.7) is sometimes called the **indicator function** of Π; that is, a function whose value is 0 on the set of interest, and $+\infty$ otherwise. And since the constraints defining Π are linear, we can write this indicator function as the solution of a supremum problem involving only linear functionals: it is easy to check that

$$\begin{cases} 0 \text{ if } \pi\in\Pi(\mu,\nu) \\ +\infty \text{ else} \end{cases} = \sup_{(\varphi,\psi)} \left[\int \varphi\,d\mu + \int \psi\,d\nu - \int [\varphi(x)+\psi(y)]\,d\pi(x,y)\right],$$

where the supremum runs over all $(\varphi,\psi)\in C_b(X)\times C_b(Y)$, for instance.

Thus the left-hand side of (1.4) is given by

$$\inf_{\pi\in M_+(X\times Y)} \sup_{(\varphi,\psi)} \left\{ \int_{X\times Y} c(x,y)\,d\pi(x,y) + \int_X \varphi\,d\mu + \int_Y \psi\,d\nu - \int_{X\times Y} [\varphi(x)+\psi(y)]\,d\pi(x,y) \right\}.$$

Taking for granted that a minimax principle can be invoked, we rewrite this as

$$\begin{aligned} &\sup_{(\varphi,\psi)} \inf_{\pi\in M_+(X\times Y)} \left\{ \int_{X\times Y} c(x,y)\,d\pi(x,y) + \int_X \varphi\,d\mu + \int_Y \psi\,d\nu - \int_{X\times Y} [\varphi(x)+\psi(y)]\,d\pi(x,y) \right\} \\ &= \sup_{(\varphi,\psi)} \left\{ \int_X \varphi\,d\mu + \int_Y \psi\,d\nu - \sup_{\pi\in M_+(X\times Y)} \int_{X\times Y} [\varphi(x)+\psi(y)-c(x,y)]\,d\pi(x,y) \right\}. \end{aligned} \tag{1.8}$$

Let us compute the supremum inside the curly brackets. If the function $\zeta(x,y) \equiv \varphi(x) + \psi(y) - c(x,y)$ takes a positive value at some point (x_0, y_0), then by choosing $\pi = \lambda\delta_{(x_0,y_0)}$ and letting $\lambda \to +\infty$ (a Dirac mass at point (x_0, y_0) with very large mass), we see that the supremum is infinite. On the other hand, if ζ is nonpositive ($d\mu \otimes d\nu$-everywhere), the supremum is clearly obtained for $\pi = 0$. Thus,

$$\sup_{\pi \in M_+(X\times Y)} \int_{X\times Y} [\varphi(x) + \psi(y) - c(x,y)]\, d\pi(x,y) = \begin{cases} 0 \text{ if } (\varphi,\psi) \in \Phi_c, \\ +\infty \text{ else.} \end{cases}$$

Plugging this into (1.8), we obtain

$$(1.8) = \sup_{(\varphi,\psi)\in\Phi_c} J(\varphi,\psi),$$

as desired.

Exercise 1.7 (Finite-dimensional linear programming). Linear programming consists in the study of the minimization or maximization of linear problems subject to inequalities defined by linear functions. One version of the standard duality relation of linear programming in $\mathbb{R}^n$ asserts that for any $b \in \mathbb{R}^m$, $c \in \mathbb{R}^n$, $A \in M_{m\times n}(\mathbb{R})$,

$$\sup_{Ax\le b} c\cdot x = \inf_{y\ge 0,\, A^T y = c} b\cdot y,$$

provided that one of these extrema is achieved; here A^T denotes the transpose of A, and the notation $y \geq 0$ means that all components of y are nonnegative,

(i) Recover this relation by a minimax principle.

(ii) (tedious!) Formally recover the Kantorovich duality as a continuous limit of it. A solution can be found in [**134**].

Remark 1.8. This connection is no accident: Kantorovich is considered as the inventor of linear programming; this was at the end of the thirties. What is more, the Monge-Kantorovich is considered as the first operational research problem to be have been solved in an algorithmic sense.

1.1.6. A rigorous minimax principle. There are several quite general theorems of convex analysis, which yield rigorous minimax principles. Following a suggestion by Brenier, we shall use here a simple but most useful one, that can be found at the very beginning of Brézis [**68**]. To state it conveniently, we introduce the concept of **Legendre-Fenchel transform**, as follows. Let E be a normed vector space, and Θ a convex function on E with values in $\mathbb{R} \cup \{+\infty\}$. Recall that the convexity assumption means

$$\forall (z_1, z_2, \lambda) \in E \times E \times [0,1], \qquad \Theta(\lambda z_1 + (1-\lambda)z_2) \le \lambda\Theta(z_1) + (1-\lambda)\Theta(z_2),$$

with the obvious extension of the operations of $\mathbb{R}$ to $\mathbb{R} \cup \{+\infty\}$. Then the Legendre-Fenchel transform of Θ is the function Θ^*, defined on the topological dual E^* of E by the formula

$$\Theta^*(z^*) = \sup_{z \in E} \big[\langle z^*, z\rangle - \Theta(z)\big].$$

Theorem 1.9 (Fenchel-Rockafellar duality). *Let E be a (separable) normed vector space, E^* its topological dual space, and Θ, Ξ two convex functions on E with values in $\mathbb{R} \cup \{+\infty\}$. Let Θ^*, Ξ^* be the Legendre-Fenchel transforms of Θ, Ξ respectively. Assume that there exists $z_0 \in E$ such that*

$$\Theta(z_0) < +\infty, \quad \Xi(z_0) < +\infty,$$

$$\Theta \textit{ is continuous at } z_0.$$

Then,

$$\inf_E \big[\Theta + \Xi\big] = \max_{z^* \in E^*} \big[-\Theta^*(-z^*) - \Xi^*(z^*)\big]. \tag{1.9}$$

Remarks 1.10. (i) It is part of the theorem that the supremum in the right-hand side above is a maximum.

(ii) The separability assumption can be removed if one is ready to use the axiom of choice.

As the reader can check, identity (1.9) is really a minimax theorem. The proof given in [**68**] is a clever but rather easy consequence of the **Hahn-Banach theorem** of separation of convex sets, a proof of which can be found in [**231, 68**] or many other sources. Proofs there use Zorn's lemma or some other version of the axiom of choice, but this can be easily dispended with under the separability assumption. We briefly sketch the argument towards Theorem 1.9, so that the reader can understand where the continuity assumption comes into play.

Proof of Theorem 1.9. What we should prove is

$$\sup_{z^* \in E^*} \inf_{x,y \in E} \Big\{\Theta(x) + \Xi(y) + \langle z^*, x - y\rangle\Big\} = \inf_{x \in E} \Big\{\Theta(x) + \Xi(x)\Big\}.$$

1. The choice $x = y$ shows that the left-hand side is not larger than the right-hand side; so we only have to prove the existence of a linear form $z^* \in E^*$ such that

$$\forall x, y \in E, \qquad \Theta(x) + \Xi(y) + \langle z^*, x - y\rangle \geq m \equiv \inf(\Theta + \Xi). \tag{1.10}$$

Since $\Theta(z_0) + \Xi(z_0) < +\infty$, the infimum m is finite.

2. Let

$$C \equiv \{(x,\lambda) \in E \times \mathbb{R}; \quad \lambda > \Theta(x)\},$$
$$C' \equiv \{(y,\mu) \in E \times \mathbb{R}; \quad \mu \leq m - \Xi(y)\}.$$

Since Θ and Ξ are convex, so are C and C'. From the assumptions in Theorem 1.9 we deduce that $(z_0, \Theta(z_0)+1) \in \mathrm{Int}(C)$, and in particular C has nonempty interior; this easily implies that $\overline{C} = \overline{\mathrm{Int}(C)}$. Moreover, C and C' are disjoint, because $m = \inf(\Theta + \Xi)$. It follows from the Hahn-Banach theorem that there exists a nontrivial linear form $\ell \in (E \times \mathbb{R})^*$ satisfying

$$\inf_{c \in C} \langle \ell, c \rangle = \inf_{c \in \mathrm{Int}(C)} \langle \ell, c \rangle \geq \sup_{c' \in C'} \langle \ell, c' \rangle.$$

In other words, there exist $w^* \in E^*$ and $\alpha \in \mathbb{R}$, $(w^*, \alpha) \neq (0,0)$, such that

$$\langle w^*, x \rangle + \alpha\lambda \geq \langle w^*, y \rangle + \alpha\mu,$$

as soon as $\lambda > \Theta(x)$ and $\mu \leq m - \Xi(y)$. As one can easily check, this is possible only if $\alpha > 0$; thus, with $z^* = w^*/\alpha$ we have

$$\langle z^*, x \rangle + \lambda \geq \langle z^*, y \rangle + \mu,$$

in particular

$$\langle z^*, x \rangle + \Theta(x) \geq \langle z^*, y \rangle + m - \Xi(y).$$

Since this holds true for all x and y in E, (1.10) is proven. □

1.1.7. Proof of the Kantorovich duality. First of all, let us list the basic properties of Polish spaces which will underlie the proof. This is just for the convenience of readers who would try to adapt the argument to a more general setting, or for those who would need some reminders about Polish spaces. A reader who is not in one of these situations may safely skip this bit.

1. First, a Borel probability measure μ on a Polish space X is automatically **regular** [**43**], which means that for any Borel set A, one has

$$\mu[A] = \sup\Big\{\mu[K]; \quad K \text{ compact, } K \subset A\Big\}$$
$$= \inf\Big\{\mu[O]; \quad O \text{ open, } A \subset O\Big\}.$$

Regularity appears naturally when one invokes the Riesz theorem [**231**, p. 40], which, roughly speaking, asserts the equivalence between nonnegative linear functionals on $C_0(X)$ and regular nonnegative measures with bounded mass when X is a locally compact Hausdorff topological space.

2. A probability measure μ on a Polish space is automatically concentrated on a σ-compact set [**43**]: there exists a measurable set S, which can be written as the union of countably many compact sets, such that $\mu[S] = 1$. Equivalently (exercise), μ is **tight**, which means that for any $\varepsilon > 0$ there

exists a compact K_ε such that $\mu[K_\varepsilon^c] \leq \varepsilon$. This result is known as **Ulam's lemma**.

3. A family $\mathcal{P}$ of probability measures on a topological space X is said to be tight if for any $\varepsilon > 0$ there exists a compact set $K_\varepsilon \subset X$ for which

$$\sup_{\mu \in \mathcal{P}} \mu[K_\varepsilon^c] \leq \varepsilon.$$

Let X be a Polish space; **Prokhorov's theorem** asserts that any tight family in $P(X)$ is relatively sequentially compact in $P(X)$: from any sequence (μ_k) in $\mathcal{P}$ one can extract a subsequence, still denoted (μ_k), and a *probability* measure μ_* on X, such that for any $\varphi \in C_b(X)$,

$$\lim_{k \to \infty} \int_X \varphi \, d\mu_k = \int_X \varphi \, d\mu_*.$$

See [**43**] for a proof; this statement can also be generalized outside the setting of Polish spaces.

4. If X is a metric space and F is a nonnegative lower semi-continuous function on X, then it can be written as the supremum of an increasing sequence of uniformly continuous nonnegative functions. To see this, just choose

$$F_n(x) = \inf_{y \in X} \bigl[F(y) + nd(x, y)\bigr], \tag{1.11}$$

where d is a metric on X, and check (excellent exercise) that the sequence (F_n) satisfies all the required properties. Note that each F_n is well-defined because F is bounded below. The property of uniform continuity will be useful to check that $\sup_{\Phi_c \cap L^1} J = \sup_{\Phi_c \cap C_b} J$.

5. If K is a compact metric space then $C(K)$ is separable (see e.g. [**127**]).

Proof of Theorem 1.3. We separate the proof into three steps, by increasing order of generality. The minimax principle will only be applied in the first step, which is the case when X and Y are compact and c is continuous. All the rest of the proof will consist in showing that this particular case implies the general statement, by approximation arguments.

1. Let us first assume that X, Y *are compact* and that c *is continuous* on $X \times Y$. Let

$$E = C_b(X \times Y)$$

be the set of all (bounded) continuous functions on $X \times Y$, equipped with its usual supremum norm $\|\cdot\|_\infty$. By Riesz' theorem, its topological dual can be identified with the space of (regular) Radon measures,

$$E^* = M(X \times Y),$$

normed by total variation. Moreover, a *nonnegative* linear form is defined by a regular nonnegative (i.e. Borel) measure.

Then we introduce

$$\Theta : u \in C_b(X \times Y) \longmapsto \begin{cases} 0 \text{ if } u(x,y) \geq -c(x,y), \\ +\infty \text{ else,} \end{cases}$$

$$\Xi : u \in C_b(X \times Y) \longmapsto \begin{cases} \displaystyle\int_X \varphi \, d\mu + \int_Y \psi \, d\nu & \text{if } u(x,y) = \varphi(x) + \psi(y), \\ \\ +\infty & \text{else.} \end{cases}$$

Note that Ξ is well-defined: if $\varphi(x) + \psi(y) = \widetilde{\varphi}(x) + \widetilde{\psi}(y)$ for all x, y, then $\varphi = \widetilde{\varphi} + s$, $\psi = \widetilde{\psi} - s$, for some $s \in \mathbb{R}$, and therefore $\int \widetilde{\varphi} \, d\mu + \int \widetilde{\psi} \, d\nu = \int \varphi \, d\mu + \int \psi \, d\nu$. The assumptions of Theorem 1.9 are obviously satisfied with $z_0 \equiv 1$, so formula (1.9) holds true.

Now, let us compute both sides of (1.9). Obviously, the left-hand side has to be

$$\inf \left\{ \int_X \varphi \, d\mu + \int_Y \psi \, d\nu; \quad \varphi(x) + \psi(y) \geq -c(x,y) \right\}$$
$$= -\sup \left\{ J(\varphi, \psi); \quad (\varphi, \psi) \in \Phi_c \right\}.$$

Next, we compute the Legendre-Fenchel transforms of Θ, Ξ. First, for any $\pi \in M(X \times Y)$,

$$\begin{aligned} \Theta^*(-\pi) &= \sup_{u \in C_b(X \times Y)} \left\{ -\int u(x,y) \, d\pi(x,y); \quad u(x,y) \geq -c(x,y) \right\} \\ &= \sup_{u \in C_b(X \times Y)} \left\{ \int u(x,y) \, d\pi(x,y); \quad u(x,y) \leq c(x,y) \right\}. \end{aligned}$$

- If π is not a nonnegative measure, then there exists a nonpositive function $v \in C_b(X \times Y)$ such that $\int v \, d\pi > 0$. Then, the choice $u = \lambda v$, with $\lambda \to +\infty$, shows that the supremum is $+\infty$.

- On the other hand, if π is nonnegative, then the supremum is clearly $\int c \, d\pi$.

Thus

$$\Theta^*(-\pi) = \begin{cases} \displaystyle\int c(x,y) \, d\pi(x,y) \text{ if } \pi \in M_+(X \times Y), \\ \\ +\infty \text{ else.} \end{cases}$$

A similar argument shows that

$$\Xi^*(\pi) = \begin{cases} 0 & \text{if} \quad \Big[\forall(\varphi,\psi) \in C_b(X) \times C_b(Y), \\ & \qquad \displaystyle\int [\varphi(x) + \psi(y)]\, d\pi(x,y) = \int \varphi\, d\mu + \int \psi\, d\nu \Big], \\ +\infty & \text{else.} \end{cases}$$

In other words, $\Theta^* - I$ and Ξ^* are the indicator functions of $M_+(X \times Y)$ and $\Pi(\mu, \nu)$, respectively. Putting everything together and changing signs, we recover

$$\inf_{\Pi(\mu,\nu)} I[\pi] = \sup_{\Phi_c \cap C_b} J(\varphi, \psi).$$

Combining this with (1.5) finishes the proof.

Exercise 1.11. Let us try to extend this proof to the non-compact case. Why would we like to replace $C_b(X \times Y)$ by $C_0(X \times Y)$? Show that if we do so in the definition of Ξ, then the latter turns out to be rather trivial, namely Ξ is the indicator function of $\{0\}$. This shows that the proof as such does not work in a non-compact setting. Still, a variation of it can be used, as we shall see later in Appendix 1.3.

2. We shall now relax the assumption of compactness; this step is the most technical. For the moment we keep the assumption that c *is bounded and uniformly continuous*. We define

$$\|c\|_\infty = \sup_{X \times Y} c(x, y).$$

We will reduce to the compact case by a careful truncation procedure. First of all, let π_* be an optimal transference plan in the Kantorovich problem, in the sense that

$$I[\pi_*] = \inf_{\pi \in \Pi(\mu,\nu)} I[\pi].$$

Since c is bounded, the infimum is obviously finite. The existence of π_* follows by the compactness of $\Pi(\mu, \nu)$, by an argument which will be detailed in Step 3. We do not give the argument here because in fact it is not essential at this point of the proof: one could work with approximate minimizers as well.

Let $\delta > 0$ be arbitrarily small. Since X and Y are Polish, so is $X \times Y$ (easy exercise). In particular, π_* is tight, and there exist compact sets $X_0 \subset X$, $Y_0 \subset Y$ such that

$$\mu[X \setminus X_0] \leq \delta, \quad \nu[Y \setminus Y_0] \leq \delta. \tag{1.12}$$

Note that, as a consequence of (1.12) (exercise),

$$\pi_*[(X \times Y) \setminus (X_0 \times Y_0)] \le 2\delta.$$

Define

$$\pi_{*0} = \frac{1_{X_0 \times Y_0}}{\pi_*[X_0 \times Y_0]}\, \pi_*;$$

note that it is a probability measure on $X_0 \times Y_0$, and let μ_0, ν_0 be the marginals of π_{*0} onto X_0, Y_0 respectively. We naturally define $\Pi_0(\mu_0, \nu_0)$ as the set of probability measures π_0 on $X_0 \times Y_0$ with marginals μ_0, ν_0, and we define I_0 on $\Pi_0(\mu_0, \nu_0)$ by

$$I_0[\pi_0] = \int_{X_0 \times Y_0} c(x,y)\, d\pi_0(x,y).$$

Let $\widetilde{\pi}_0 \in \Pi_0(\mu_0, \nu_0)$ be such that

$$I_0[\widetilde{\pi}_0] = \inf_{\pi_0 \in \Pi_0(\mu_0,\nu_0)} I_0[\pi_0].$$

From $\widetilde{\pi}_0$ we construct a $\widetilde{\pi} \in \Pi(\mu, \nu)$ in a natural way, by gluing together $\widetilde{\pi}_0$ with a little bit of π_*:

$$\widetilde{\pi} = \pi_*[X_0 \times Y_0]\, \widetilde{\pi}_0 + 1_{(X_0 \times Y_0)^c} \pi_*$$

(check that $\widetilde{\pi} \in \Pi(\mu, \nu)$!).

So

$$\begin{aligned} I[\widetilde{\pi}] &= \pi_*[X_0 \times Y_0]\, I_0[\widetilde{\pi}_0] + \int_{(X_0 \times Y_0)^c} c(x,y)\, d\pi_*(x,y) \\ &\le I_0[\widetilde{\pi}_0] + 2\|c\|_\infty \delta = \inf I_0 + 2\|c\|_\infty \delta. \end{aligned}$$

It follows that

$$\inf_{\pi \in \Pi(\mu,\nu)} I[\pi] \le \inf I_0 + 2\|c\|_\infty \delta.$$

(Note: It was natural to introduce the minimizer π_*, whose existence we knew; but actually we could have arrived at the same conclusion by showing that for any plan π on $X_0 \times Y_0$ there is $\widetilde{\pi}$ on $X \times Y$ such that $|I[\pi] - I_0[\pi_0]| \le 2\|c\|_\infty\, \delta$.)

Now introduce the functional

$$J_0(\varphi_0, \psi_0) = \int_{X_0} \varphi_0\, d\mu_0 + \int_{Y_0} \psi_0\, d\nu_0,$$

defined on $L^1(d\mu_0) \times L^1(d\nu_0)$. By Step 1 of the proof, we know that $\inf I_0 = \sup J_0$, where the supremum runs over all admissible couples $(\varphi_0, \psi_0) \in L^1(d\mu_0) \times L^1(d\nu_0)$, i.e. those which satisfy $\varphi_0(x) + \psi_0(y) \le c(x,y)$ for almost all x, y. In particular, there exist an admissible couple of functions $\widetilde{\varphi}_0$, $\widetilde{\psi}_0$ such that

$$J_0(\widetilde{\varphi}_0, \widetilde{\psi}_0) \ge \sup J_0 - \delta.$$

Our problem is to construct from $(\widetilde{\varphi}_0, \widetilde{\psi}_0)$ a couple (φ, ψ) which would be very efficient in the problem of maximization of $J(\varphi, \psi)$.

It will be useful to ensure that the inequality $\widetilde{\varphi_0}(x) + \widetilde{\psi_0}(y) \leq c(x, y)$ is valid for *all* x and y, not just almost all. We can ensure this, provided that we allow $\widetilde{\varphi_0}$ and $\widetilde{\psi_0}$ to take values in $\mathbb{R} \cup \{-\infty\}$. Indeed, we can introduce negligible sets N_x and N_y such that the inequality holds true for $(x, y) \in N_x^c \times N_y^c$, and redefine the values of φ, ψ to be $-\infty$ on N_x, N_y respectively.

In a first step, we will control $\widetilde{\varphi}_0$, $\widetilde{\psi}_0$ from below at some point in $X \times Y$. Without loss of generality, we assume that $\delta \leq 1$. Since $J_0(0, 0) = 0$, we know that $\sup J_0 \geq 0$, and hence $J_0(\widetilde{\varphi}_0, \widetilde{\psi}_0) \geq -\delta \geq -1$. By writing

$$J_0(\widetilde{\varphi}_0, \widetilde{\psi}_0) = \int_{X \times Y} [\widetilde{\varphi}_0(x) + \widetilde{\psi}_0(y)] \, d\pi_0(x, y),$$

where π_0 is any element of $\Pi_0(\mu_0, \nu_0)$, we deduce that there exists $(x_0, y_0) \in X_0 \times Y_0$ such that

$$\widetilde{\varphi}_0(x_0) + \widetilde{\psi}_0(y_0) \geq -1. \tag{1.13}$$

If we replace $(\widetilde{\varphi}_0, \widetilde{\psi}_0)$ by $(\widetilde{\varphi}_0 + s, \widetilde{\psi}_0 - s)$ for some real number s, we do not change the value of $J_0(\widetilde{\varphi}_0, \widetilde{\psi}_0)$ and the resulting couple is still admissible. By a proper choice of s, we can ensure

$$\widetilde{\varphi}_0(x_0) \geq -\frac{1}{2}, \qquad \widetilde{\psi}_0(y_0) \geq -\frac{1}{2}.$$

This implies that, for all $(x, y) \in X_0 \times Y_0$,

$$\widetilde{\varphi}_0(x) \leq c(x, y_0) - \widetilde{\psi}_0(y_0) \leq c(x, y_0) + \frac{1}{2},$$

$$\widetilde{\psi}_0(y) \leq c(x_0, y) - \widetilde{\varphi}_0(x_0) \leq c(x_0, y) + \frac{1}{2}.$$

To go further we shall use a key trick for "improving" admissible pairs of functions, which was popularized by Rüschendorf. It will be encountered again several times in this book. Define, for $x \in X$,

$$\overline{\varphi}_0(x) = \inf_{y \in Y_0} [c(x, y) - \widetilde{\psi}_0(y)].$$

From the inequality $\widetilde{\varphi}_0(x) \leq c(x, y) - \widetilde{\psi}_0(y)$ we see that $\widetilde{\varphi}_0 \leq \overline{\varphi}_0$ on X_0. This implies $J_0(\overline{\varphi}_0, \widetilde{\psi}_0) \geq J_0(\widetilde{\varphi}_0, \widetilde{\psi}_0)$. Moreover, for all $x \in X$ we have a control of $\overline{\varphi}_0(x)$ from above and below, in terms of the cost function:

$$\overline{\varphi}_0(x) \geq \inf_{y \in Y_0} [c(x, y) - c(x_0, y)] - \frac{1}{2},$$

$$\overline{\varphi}_0(x) \leq c(x, y_0) - \widetilde{\psi}_0(y_0) \leq c(x, y_0) + \frac{1}{2}.$$

Finally we define, for $y \in Y$,

$$\overline{\psi}_0(y) = \inf_{x \in X} [c(x,y) - \overline{\varphi}_0(x)],$$

and we still have $(\overline{\varphi}_0, \overline{\psi}_0) \in \Phi_c$. Then, it is straightforward to check that $J_0(\overline{\varphi}_0, \overline{\psi}_0) \geq J_0(\overline{\varphi}_0, \widetilde{\psi}_0) \geq J_0(\widetilde{\varphi}_0, \widetilde{\psi}_0)$. Moreover, for all $y \in Y$,

$$\overline{\psi}_0(y) \geq \inf_{x \in X} [c(x,y) - c(x,y_0)] - \frac{1}{2},$$

$$\overline{\psi}_0(y) \leq c(x_0,y) - \overline{\varphi}_0(x_0) \leq c(x_0,y) - \widetilde{\varphi}_0(x_0) \leq c(x_0,y) + \frac{1}{2}.$$

In particular,

$$\overline{\varphi}_0(x) \geq -\|c\|_\infty - \frac{1}{2},$$

$$\overline{\psi}_0(y) \geq -\|c\|_\infty - \frac{1}{2}.$$

Once we have these bounds, we are almost done! Indeed,

$$J(\overline{\varphi}_0, \overline{\psi}_0) = \int_X \overline{\varphi}_0 \, d\mu + \int_Y \overline{\psi}_0 \, d\nu = \int_{X \times Y} \left[\overline{\varphi}_0(x) + \overline{\psi}_0(y)\right] d\pi_*(x,y)$$

$$= \pi_*[X_0 \times Y_0] \int_{X_0 \times Y_0} \left[\overline{\varphi}_0(x) + \overline{\psi}_0(y)\right] d\pi_{*0}(x,y)$$

$$+ \int_{(X_0 \times Y_0)^c} \left[\overline{\varphi}_0(x) + \overline{\psi}_0(y)\right] d\pi_*(x,y)$$

$$\geq (1 - 2\delta) \left(\int_{X_0} \overline{\varphi}_0 \, d\mu_0 + \int_{Y_0} \overline{\psi}_0 \, d\nu_0 \right) - (2\|c\|_\infty + 1)\, \pi_*[(X_0 \times Y_0)^c]$$

$$\geq (1 - 2\delta)\, J_0(\overline{\varphi}_0, \overline{\psi}_0) - 2(2\|c\|_\infty + 1)\delta$$

$$\geq (1 - 2\delta)\, J_0(\widetilde{\varphi}_0, \widetilde{\psi}_0) - 2(2\|c\|_\infty + 1)\delta$$

$$\geq (1 - 2\delta)(\inf I_0 - \delta) - 2(2\|c\|_\infty + 1)\delta$$

$$\geq (1 - 2\delta)\big(\inf I - (2\|c\|_\infty + 1)\delta\big) - 2(2\|c\|_\infty + 1)\delta.$$

Since δ is arbitarily small, we conclude that $\sup J(\varphi, \psi) \geq \inf I$, which was our goal.

Note that the functions $\overline{\varphi}_0$, $\overline{\psi}_0$ are continuous (and even uniformly continuous) on the whole of X, Y since c is uniformly continuous. Therefore, it does not matter whether we take the supremum of J over $\Phi_c \cap L^1$ or over $\Phi_c \cap C_b$. By the way, this also shows that $\overline{\varphi}_0$ and $\overline{\psi}_0$ are measurable, a property which otherwise would not be obvious to establish.

3. We finally turn to the general case. Write $c = \sup c_n$, where c_n is a nondecreasing sequence of nonnegative, uniformly continuous cost functions. Upon replacing c_n by $\inf(c_n, n)$, one can assume that each c_n is bounded.

Now let I_n be defined on $\Pi(\mu,\nu)$ by

$$I_n[\pi] = \int_{X\times Y} c_n\, d\pi.$$

From Step 2 we know that

$$\inf_{\pi\in\Pi(\mu,\nu)} I_n[\pi] = \sup_{(\varphi,\psi)\in\Phi_{c_n}} J(\varphi,\psi). \tag{1.14}$$

We will conclude the argument by showing that

$$\inf_{\pi\in\Pi(\mu,\nu)} I[\pi] = \sup_n \inf_{\pi\in\Pi(\mu,\nu)} I_n[\pi] \tag{1.15}$$

and that, for each n,

$$\sup_{(\varphi,\psi)\in\Phi_{c_n}} J(\varphi,\psi) \le \sup_{(\varphi,\psi)\in\Phi_c} J(\varphi,\psi). \tag{1.16}$$

Indeed, the combination of (1.14), (1.15) and (1.16) will imply that

$$\inf_{\pi\in\Pi(\mu,\nu)} I[\pi] \le \sup_{(\varphi,\psi)\in\Phi_c} J(\varphi,\psi),$$

and we already know that the reverse inequality is always true.

Since $c_n \le c$ by construction, it follows that Φ_{c_n} is a subset of Φ_c, on which J_n coincides with J, so (1.16) is trivial. Moreover, here it does not matter whether we define Φ_c and Φ_{c_n} as subsets of $C_b(X)\times C_b(Y)$, or as subsets of $L^1(d\mu)\times L^1(d\nu)$.

Since I_n is a nondecreasing sequence of functionals, it is also clear that $\inf I_n$ is a nondecreasing sequence, bounded above by $\inf I$. Therefore, we only have to prove that

$$\lim_{n\to\infty} \inf_{\pi\in\Pi(\mu,\nu)} I_n[\pi] \ge \inf_{\pi\in\Pi(\mu,\nu)} I[\pi]. \tag{1.17}$$

Next, we note that the set $\Pi(\mu,\nu)$ is tight. Indeed, since μ and ν are tight, for any $\varepsilon>0$ there exist compact sets $K_\varepsilon\subset X$ and $L_\varepsilon\subset Y$ such that

$$\mu[K_\varepsilon^c]\le\varepsilon/2, \qquad \nu[L_\varepsilon^c]\le\varepsilon/2;$$

then for any $\pi\in\Pi(\mu,\nu)$,

$$\pi[(K_\varepsilon\times L_\varepsilon)^c] \le \pi[K_\varepsilon\times Y] + \pi[X\times L_\varepsilon] = \mu[K_\varepsilon]+\nu[L_\varepsilon]\le\varepsilon.$$

By Prokhorov's theorem, this implies that $\Pi(\mu,\nu)$ is relatively compact for the weak topology. In particular, if $(\pi_n^k)_{k\in\mathbb{N}}$ is any minimizing sequence for the problem $\inf I_n[\pi]$, then we know that, up to extraction of a subsequence, π_n^k converges weakly to some probability measure $\pi_n\in P(X\times Y)$ as $k\to\infty$, in the sense that for any bounded continuous function θ on $X\times Y$,

$$\int \theta(x,y)\, d\pi_n^k(x,y) \xrightarrow[k\to\infty]{} \int \theta(x,y)\, d\pi_n(x,y).$$

From this we immediately see that π_n belongs to $\Pi(\mu,\nu)$ and that

$$\inf I_n = \lim_{k\to\infty} \int c_n \, d\pi_n^k = \int c_n \, d\pi_n,$$

which shows the existence of a minimizing probability measure π_n.

Similarly, the sequence $(\pi_n)_{n\in\mathbb{N}}$ admits a cluster point π_* by compactness of $\Pi(\mu,\nu)$. Whenever $n \geq m$, one has

$$I_n[\pi_n] \geq I_m[\pi_n].$$

So, by continuity of I_m,

$$\lim_{n\to\infty} I_n[\pi_n] \geq \limsup_{n\to\infty} I_m[\pi_n] \geq I_m[\pi_*].$$

By monotone convergence, $I_m[\pi_*] \longrightarrow I[\pi_*]$ as $m \to \infty$, so

$$\lim_{n\to\infty} I_n[\pi_n] \geq \lim_{m\to\infty} I_m[\pi_*] = I[\pi_*] \geq \inf_{\pi\in\Pi(\mu,\nu)} I[\pi],$$

which proves (1.17), and concludes the proof of (1.4).

4. To complete the proof of Theorem 1.3, it only remains to check that the infimum is attained. This is again a consequence of the compactness of $\Pi(\mu,\nu)$. Indeed, let (π_k) be a minimizing sequence for I, and let π_* be any weak cluster point of (π_k); then, by invoking the monotone convergence theorem for the increasing sequence (c_n), we have

$$I[\pi_*] = \lim_{n\to\infty} I_n[\pi_*] \leq \lim_{n\to\infty} \limsup_{k\to\infty} I_n[\pi_k] \leq \limsup_{k\to\infty} I[\pi_k] = \inf I.$$

□

Remark 1.12 (c-concave functions). It follows from the proof that, when c is bounded, one can restrict the supremum in the right-hand side of (1.4) to those pairs $(\varphi^{cc}, \varphi^c)$, where φ is bounded and

$$\varphi^c(y) = \inf_{x\in X} [c(x,y) - \varphi(x)], \qquad \varphi^{cc}(x) = \inf_{y\in Y} [c(x,y) - \varphi^c(y)]. \tag{1.18}$$

An easy argument shows that $(\varphi^{cc})^c = \varphi^c$ (see Exercise 2.35). The pair $(\varphi^{cc}, \varphi^c)$ is called a pair of conjugate c-**concave functions**. Note that φ^c is measurable, since it can be written (exercise) as $\lim_{\ell\to\infty} \psi_\ell$, where

$$\psi_\ell(y) = \inf_{x\in X} [c_\ell(x,y) - \varphi(x)],$$

and c_ℓ is an increasing family of bounded uniformly continuous functions converging pointwise to c. Indeed, each ψ_ℓ is uniformly continuous, and therefore φ^c is measurable. Similarly, φ^{cc} is measurable.

One can give an alternative derivation of the Kantorovich duality via the study of c-concave functions, see Remark 2.40. Although this notion is elementary, we prefer to develop it only in Chapter 2, after some reminders about the classical duality of convex functions.

Remark 1.13 (Estimates for bounded cost functions). In the case when c is bounded, it is sometimes useful to know that the supremum can be further restricted:

$$\begin{aligned}&\sup\Big\{J(\varphi,\psi);\ (\varphi,\psi)\in\Phi_c\Big\}\\&\quad=\sup\Big\{J(\varphi,\psi);\quad(\varphi,\psi)\in\Phi_c,\ 0\le\varphi\le\|c\|_\infty,\ -\|c\|_\infty\le\psi\le 0\Big\}.\end{aligned}$$

This is a consequence of Remark 1.12 and the following estimates on pairs of conjugate c-concave functions:

$$\left\{\begin{array}{ccccc}-\sup\varphi & \le & \varphi^c & \le & \|c\|_\infty-\sup\varphi,\\ -\sup\varphi^c & \le & \varphi=\varphi^{cc} & \le & \|c\|_\infty-\sup\varphi^c.\end{array}\right. \tag{1.19}$$

Since $J(\varphi+s,\psi-s)=J(\varphi,\psi)$ for all $s\in\mathbb{R}$, and $(\varphi+s)^c=\varphi^c-s$, we can assume without loss of generality that $\sup\varphi=\|c\|_\infty$. Then it follows from (1.19) that $-\|c\|_\infty\le\varphi^c\le 0$, and this in turns implies $\inf\varphi\ge 0$.

1.2. Distance cost functions

When the cost function is a metric: $c(x,y)=d(x,y)$ on $X=Y$, then there is more structure in the Kantorovich duality principle. Note carefully that this distance need not be the distance defining the topology of the space.

1.2.1. The Kantorovich-Rubinstein theorem.

Theorem 1.14 (Kantorovich-Rubinstein theorem). *Let $X=Y$ be a Polish space and let d be a lower semi-continuous metric on X. Let $\mathcal{T}_d$ be the cost of optimal transportation for the cost $c(x,y)=d(x,y)$,*

$$\mathcal{T}_d(\mu,\nu)=\inf_{\pi\in\Pi(\mu,\nu)}\int_{X\times X}d(x,y)\,d\pi(x,y).$$

Let $\mathrm{Lip}(X)$ *denote the space of all Lipschitz functions on X, and*

$$\|\varphi\|_{\mathrm{Lip}}\equiv\sup_{x\neq y}\frac{|\varphi(x)-\varphi(y)|}{d(x,y)}.$$

Then

$$\mathcal{T}_d(\mu,\nu)=\sup\left\{\int_X\varphi\,d(\mu-\nu);\quad\varphi\in L^1(d|\mu-\nu|),\quad\|\varphi\|_{\mathrm{Lip}}\le 1\right\}.$$

Moreover, it does not change the value of the supremum above to impose the additional condition that φ be bounded.

Remarks 1.15. (i) When d is bounded, it is even possible to restrict the supremum to those φ's satisfying $0\le\varphi\le\|d\|_\infty$.

(ii) Let $P_1(X)$ be the space of those probability measures μ such that $\int d(x_0, x)\, d\mu(x) < +\infty$ for some (and thus any) x_0, and let $M_1(X)$ be the vector space generated by $P_1(X)$. On $M_1(X)$ we can define the norm

$$\|\sigma\|_{KR} = \sup\left\{ \int_X \varphi\, d\sigma; \quad \varphi \in L^1(d|\sigma|), \quad \|\varphi\|_{\mathrm{Lip}} \leq 1 \right\}. \tag{1.20}$$

Then the Kantorovich-Rubinstein theorem states that $\mathcal{T}_d(\mu, \nu) = \|\mu - \nu\|_{KR}$ for all probability measures μ, ν in $P_1(X)$.

(iii) When d is bounded, $\mathcal{T}_d$ and $\|\cdot\|_{KR}$ are well-defined on $P(X)$ and $M(X)$ respectively; then $\mathcal{T}_d$ is sometimes called the "bounded Lipschitz distance", and $\|\cdot\|_{KR}$ the "bounded Lipschitz norm". However it is more standard to define the "bounded Lipschitz distance" in a slightly different way [**127**]:

$$d_{BL^*}(\mu, \nu) = \|\mu - \nu\|_{BL^*} \equiv \sup_{\|\varphi\|_{BL} \leq 1} \int \varphi\, d(\mu - \nu),$$

where $\|\varphi\|_{BL} \equiv \|\varphi\|_\infty + \|\varphi\|_{\mathrm{Lip}}$. Obviously, $\|\cdot\|_{BL^*} \leq \|\cdot\|_{KR}$, but there is no equality in general: in particular, $\|\cdot\|_{BL^*}$ is defined on the whole of $M(X)$ even when d is unbounded. For sequences of probability measures, convergence in distance d_{BL^*} is equivalent to weak convergence, while convergence in distance $\mathcal{T}_d$ is just slightly stronger when X is unbounded, as we shall discuss in Chapter 7.

From the Kantorovich-Rubinstein theorem we will now deduce a corollary which is intuitively clear, but not so trivial to prove. To state it in a clear way, it will be convenient to consider mass transportation between nonnegative measures whose mass is not normalized to unity (this extension is straightforward).

Corollary 1.16 (Invariance of Kantorovich-Rubinstein distance under mass subtraction). *Let $X = Y$ be a Polish space and let d be a lower semi-continuous distance on X. Let μ, ν and σ be three Borel (nonnegative) measures on X, such that $\mu[X] = \nu[X] < +\infty$, $\sigma[X] < +\infty$. Then*

$$\mathcal{T}_d(\mu + \sigma, \nu + \sigma) = \mathcal{T}_d(\mu, \nu).$$

Of course the bound $\mathcal{T}_d(\mu + \sigma, \nu + \sigma) \leq \mathcal{T}_d(\mu, \nu)$ is immediate (exercise); however the converse bound would be more tricky to establish! This corollary can be reformulated in the following way: whenever μ and ν are two probability measures on X, then

$$\mathcal{T}_d(\mu, \nu) = \mathcal{T}_d\Big((\mu - \nu)_+, (\nu - \mu)_+\Big).$$

In other words, in a mass transportation problem with a distance cost function, one can assume that all the mass shared between the two probability measures does stay in place. In the above the notation ρ_+ stands for the

nonnegative part of the Radon measure ρ; it is defined by the characteristic property that ρ can be written as the Hahn decomposition $\rho_+ - \rho_-$, where ρ_+ and ρ_- are nonnegative Borel measures which are *singular to each other*, i.e. concentrated on disjoint sets.

We now come to the proof of the Kantorovich-Rubinstein theorem. We note that a more general version (without the completeness assumption) is proven in [**127**, Section 11.8].

Proof of Theorem 1.14. Let $d_n = d/(1+n^{-1}d)$: this is a distance satisfying $d_n \leq d$, and for all x, y the quantity $d_n(x,y)$ converges monotonically to $d(x,y)$ as $n \to \infty$. In particular, the set of 1-Lipschitz functions for d_n is included in the set of 1-Lipschitz functions for d. Reasoning as in Step 3 of the proof of Theorem 1.3, we see that we just have to prove Theorem 1.14 with d replaced by d_n. Hence, in the sequel we shall assume that d is bounded. In this case, all Lipschitz functions are bounded, and therefore integrable with respect to μ, ν. So, in view of Theorem 1.3, the only thing to check is that

$$\sup_{(\varphi,\psi)\in\Phi_d} J(\varphi,\psi) = \sup\left\{\int_X \varphi\, d(\mu-\nu); \quad \|\varphi\|_{\mathrm{Lip}} \leq 1\right\},$$

where $J(\varphi,\psi) = \int_X \varphi\, d\mu + \int_Y \psi\, d\nu$.

From Remark 1.12 we know that

$$\sup_{(\varphi,\psi)\in\Phi_d} J(\varphi,\psi) = \sup_{\varphi\in L^1(d\mu)} J(\varphi^{dd},\varphi^d),$$

where

$$\varphi^d(y) \equiv \inf_{x\in X}[d(x,y)-\varphi(x)], \qquad \varphi^{dd}(x) \equiv \inf_{y\in X}[d(x,y)-\varphi^d(y)].$$

Now, φ^d, being an infimum of 1-Lipschitz functions, bounded from below at some point x_0, is 1-Lipschitz. So

$$-\varphi^d(x) \leq \inf_y[d(x,y)-\varphi^d(y)] \leq -\varphi^d(x),$$

where the right inequality follows by the choice $x = y$ in the infimum, and the left inequality by the 1-Lipschitz property. This means that $\varphi^{dd} = -\varphi^d$, and

$$\begin{aligned}\sup_{\Phi_c} J(\varphi,\psi) \leq \sup_{\varphi\in L^1(d\mu)} J(\varphi^{dd},\varphi^d) &= \sup_{\varphi\in L^1(d\mu)} J(-\varphi^d,\varphi^d) \\ &\leq \sup_{\|\varphi\|_{\mathrm{Lip}}\leq 1} J(\varphi,-\varphi) \leq \sup_{\Phi_c} J(\varphi,\psi).\end{aligned}$$

So there is equality everywhere, and the result follows. □

Exercise 1.17 (Total variation formula). Let X be a Polish space. Show that the assumptions of Theorem 1.14 are satisfied with the cost function $c(x,y) = 1_{x\neq y}$. Use this to show that whenever μ and ν are Borel probability measures on X,

$$\inf_{\pi\in\Pi(\mu,\nu)} \pi[\{x \neq y\}] = \sup_{0\le f\le 1} \int_X f\, d(\mu-\nu).$$

Using the decomposition $(\mu-\nu) = (\mu-\nu)_+ - (\mu-\nu)_-$, where $(\mu-\nu)_\pm$ are singular to each other, show that

$$\sup_{0\le f\le 1} \int_X f\, d(\mu-\nu) = (\mu-\nu)_+[X] = (\mu-\nu)_-[X] = \frac{1}{2}\|\mu-\nu\|_{TV}.$$

Using the regularity of $(\mu-\nu)_+$, show that in fact

$$(\mu-\nu)_+[X] = \sup_K (\mu-\nu)_+[K] = \sup_K (\mu[K]-\nu[K]),$$

where the suprema are taken over all compact subsets K of X.

Exercise 1.18 (Dual of $C^{0,\alpha}$). Let Ω be a smooth bounded open subset of $\mathbb{R}^n$, let $\alpha \in (0,1)$ and let $E = C^{0,\alpha}(\Omega)$ be the Banach space of α-Hölder continuous real-valued functions on Ω, equipped with the norm

$$\|f\|_E = \sup|f(x)| + \sup_{x\neq y} \frac{|f(x)-f(y)|}{|x-y|^\alpha}.$$

Of course one can view the space $M(\Omega)$ as a subset of E^*, equipped with the norm

$$\|\ell\|_{E^*} = \sup_{\|f\|_E\le 1} \langle \ell, f\rangle.$$

Let μ and ν be two probability measures in E^*; show that

$$\|\mu-\nu\|_{E^*} \le \min\big(\|\mu-\nu\|_{TV}, \mathcal{T}_c(\mu,\nu)\big) \tag{1.21}$$

where $c(x,y) = |x-y|^\alpha$.

Remark 1.19. It is interesting to note that $\mathcal{T}_c$ above defines a norm which, roughly speaking, induces the topology of weak convergence of measures. We shall examine this in Chapter 7, and make precise statements there.

1.2.2. Transshipment. The Kantorovich-Rubinstein theorem implies that the total cost only depends on the difference $\mu-\nu$. Thus, *when the cost function is a metric*, the Kantorovich optimal transportation problem is equivalent to the **Kantorovich-Rubinstein transshipment problem**:

$$\inf\big\{I[\pi];\quad \pi[A\times X] - \pi[X\times A] = (\mu-\nu)[A]\big\}.$$

The condition appearing above should be compared to the condition for $\pi \in \Pi(\mu,\nu)$, which is $\pi[A\times X] = \mu[A]$, $\pi[X\times A] = \nu[A]$. For a general cost function, the transshipment problem is a strongly relaxed version of

the transportation problem. For instance, in the case of a quadratic cost in $\mathbb{R}^n$, the optimal transshipment cost between two given measures is in general 0. We shall *not* study the transshipment problem in this course, and refer to [**225**] for motivations and detailed study.

Exercise 1.20. Give an interpretation of the Kantorovich-Rubinstein transshipment problem in (say) economics terms; contrast this interpretation with that of the Monge-Kantorovich problem.

Exercise 1.21 (Transshipment sometimes costs (almost) nothing). Let $c(x,y) = |x-y|^2$ in $\mathbb{R}^n$, and let μ, ν be two probability measures on $\mathbb{R}^n$, such that $\mathcal{T}_c(\mu,\nu) < +\infty$. Let $\pi \in \Pi(\mu,\nu)$ be any transference plan such that $I[\pi] < +\infty$. Of course this transference plan can also be considered as a transshipment plan, with an associated transshiping cost. In order to lower this transshipment cost, you wish to improve this plan, and you come up with the following strategy. Whenever x and y are some initial and final points, respectively, instead of shipping x to y, you ship x to $(x+y)/2$ and *simultaneously* $(x+y)/2$ to y. Show that this strategy can be implemented with an admissible transshipment plan, and express it in terms of image measures of π. Make a schematic picture of how π is modified by this transformation. Show that the cost has been lowered by a factor 2. Deduce that the optimal transshipment cost is 0, which of course is not attained unless $\mu = \nu$. Show that the conclusion is still valid for any power $|x-y|^p$, $p > 1$, or more generally as soon as $c(x,y) = \phi(|x-y|)$, where ϕ is nondecreasing on $\mathbb{R}_+$, $\phi(0) = 0$, $\phi'(0) = 0$.

1.2.3. Divergence formulation. There is yet another way to rewrite the Kantorovich-Rubinstein theorem in $\mathbb{R}^n$, when the distance d is the standard Euclidean distance. Indeed, in this case, the condition $\|\varphi\|_{\mathrm{Lip}} \leq 1$ can also be rewritten as

$$\|\nabla\varphi\|_{L^\infty} \leq 1.$$

By a new minimax argument which is left as an exercise, we find a new dual formulation (well, dual of the dual):

$$\mathcal{T}_d(\mu,\nu) = \inf\big\{\|\sigma\|_{L^1(\mathbb{R}^n)};\quad \nabla\cdot\sigma = \mu - \nu\big\}. \tag{1.22}$$

Here $\nabla\cdot$ denotes the divergence operator, acting on $\mathcal{D}'(\mathbb{R}^n)$. This formulation is very useful for numerical simulations ("minimal network flow problem"), but also for theoretical purposes, as we shall explain later on. A similar formulation would be available on Riemannian manifolds, with a cost function equal to the geodesic distance.

1.3. Appendix: A duality argument in $C_b(X \times Y)$

In Section 1.1 we have used the Fenchel-Rockafellar duality theorem only in a particular case, when the underlying spaces X and Y were compact; the rest of the proof of Theorem 1.3 was done by approximation. What happens if we try to directly apply the duality theorem in a non-compact case? This will not lead us to any improvement of Theorem 1.3, but will be rather instructive. To work out the method, we shall restrict the generality of the assumptions a little bit. Certainly variants of the proof would work in a more general setting (as suggested by some of the arguments in [**225**]), but since we already gave a general enough statement in Theorem 1.3, we shall not look for such extensions.

Note carefully: This Appendix uses the axiom of choice, through the Hahn-Banach theorem in a nonseparable space. This is the only place in the book where the axiom of choice really plays a role.

Proposition 1.22 (Kantorovich duality again). *Let X and Y be locally compact Polish spaces, let c be a lower semi-continuous nonnegative function on $X \times Y$, and let μ, ν be two Borel probability measures on X, Y respectively. Then*

$$\inf_{\pi \in \Pi(\mu,\nu)} I[\pi] = \sup_{(\varphi,\psi)\in\Phi_c} J(\varphi,\psi). \tag{1.23}$$

Let us recall again that a topological space is locally compact if any of its elements admits a compact neighborhood. All the notions which we shall need about such spaces, in particular **Urysohn's extension lemma**, can be found in Rudin [**231**].

The idea of the proof is to copy Step 1 in the proof of Theorem 1.3. We would now like to choose $E = C_0(X \times Y)$, because then E^* would be $M(X \times Y)$. But if we do so, then Ξ becomes trivial, as noticed in Exercise 1.11. We therefore choose $E = C_b(X \times Y)$. The tricky point is that if $X \times Y$ is non-compact, then E^* is larger than $M(X \times Y)$. Of course, if $\ell \in E^*$, then ℓ acts continuously on the subset $C_0(X \times Y)$ of $C_b(X \times Y)$, so there exists a unique $\pi = \pi(\ell) \in M(X \times Y)$ such that

$$\forall u \in C_0(X \times Y), \quad \langle \ell, u\rangle = \int_{X\times Y} u\, d\pi.$$

Thus we can write $\ell = \pi + R$, where R is a continuous linear functional supported at infinity, in the sense that

$$u \in C_0(X \times Y) \Longrightarrow \langle R, u\rangle = 0.$$

As the following exercise shows, some of these R's may be weird, and we should be careful when handling them.

Exercise 1.23 (Be cautious when working in $(C_b)^*$). We use the same notation as above. A function $u \in C_b(X)$ is said to admit a limit $u(\infty)$ at infinity, if for any $\varepsilon > 0$ there exists a compact $K_\varepsilon \subset X$ such that $x \notin K_\varepsilon \Longrightarrow |u(x) - u(\infty)| \leq \varepsilon$.

(i) Use the Hahn-Banach extension theorem to construct a continuous extension of the linear functional "limit at infinity". Show that this extension is supported at infinity. Note that it attributes a "limit" to *any* bounded continuous function. Of course this needs the axiom of choice.

(ii) Let $\mu \in P(X)$, $\nu \in P(Y)$ be Borel probability measures. Construct a continuous linear functional ℓ on $C_b(X \times Y)$, *supported at infinity*, such that

$$\forall (\varphi, \psi) \in C_0(X) \times C_0(Y), \qquad \langle \ell, \varphi + \psi \rangle = \int_X \varphi \, d\mu + \int_Y \psi \, d\nu. \tag{1.24}$$

Here we used the notation $\langle \cdot, \cdot \rangle$ to denote the duality bracket between E and E^*, and the shorthand $(\varphi + \psi)(x, y) = \varphi(x) + \psi(y)$. Note that, as a consequence of this exercise, equation (1.24) does not at all guarantee that $\ell \in \Pi(\mu, \nu)$!

Hint: Think of something like

$$\int_X \left[\lim_{y \to \infty} u(x, y) \right] d\mu(x) + \int_Y \left[\lim_{x \to \infty} u(x, y) \right] d\nu(y).$$

There are representation theorems for C_b^*, in the form of *finitely* additive (as opposed to σ-additive) measures, but we prefer to avoid using them. The following two lemmas will be enough to keep us on safe ground. The notation is the same as above; we recall that a σ-compact set is a countable union of compact sets.

Lemma 1.24 (Decomposition of nonnegative elements in $(C_b)^*$). *Let X and Y be locally compact, σ-compact Polish spaces. Let ℓ be a nonnegative linear form on $C_b(X \times Y)$. Then it can be written as $\pi + R$, where π is a nonnegative measure and R a nonnegative continuous linear functional supported at infinity.*

Lemma 1.25 (Marginal condition in $(C_b)^*$). *Let X and Y be locally compact, σ-compact Polish spaces, and let $\mu \in P(X)$, $\nu \in P(Y)$ be Borel probability measures. Let ℓ be a nonnegative linear form on $C_b(X \times Y)$ such that, for all bounded continuous functions φ on X and ψ on Y,*

$$\langle \ell, \varphi + \psi \rangle = \int_X \varphi \, d\mu + \int_Y \psi \, d\nu. \tag{1.25}$$

Then ℓ is a nonnegative measure, and in particular is an element of $\Pi(\mu, \nu)$.

Remark 1.26. Let ℓ be a nonnegative linear form on $C_b(X \times Y)$. Thanks to Lemma 1.24, we can write $\ell = \pi + R$, where both π and R are nonnegative, and deduce that for any pair (φ, ψ) of *nonnegative* functions in $C_b(X) \times C_b(Y)$,

$$\int_{X \times Y} [\varphi(x) + \psi(y)]\, d\pi(x, y) \leq \int_X \varphi\, d\mu + \int_Y \psi\, d\nu. \tag{1.26}$$

But this alone does not immediately imply that $\pi \in \Pi(\mu, \nu)$, as shown by Exercise 1.23. The proof of Lemma 1.25 will in fact use in an essential way the identity $\langle \ell, 1 \rangle = 1$.

Proof of Lemma 1.24. Let us write $\ell = \pi + R$. Since ℓ is nonnegative, so is its restriction π to $C_0(X \times Y)$. So we just have to prove that $\ell \geq \pi$ on $C_b(X \times Y)$, in the sense of linear functionals. Let $u(x, y)$ be any continuous bounded nonnegative function, and let $\chi_n(x, y)$ be an increasing sequence of continuous cut-off functions with compact support, $0 \leq \chi_n(x, y) \leq 1$, such that for all $(x, y) \in X \times Y$, $\chi_n(x, y) \longrightarrow 1$ as $n \to \infty$. The existence of the sequence (χ_n) follows from the assumption of σ-compactness and Urysohn's lemma. Then we can write

$$\int_{X \times Y} u\, d\pi = \lim_{n \to \infty} \int_{X \times Y} u \chi_n\, d\pi = \lim_{n \to \infty} \langle \ell, u \chi_n \rangle \leq \langle \ell, u \rangle,$$

where the first equality is a consequence of the monotone convergence theorem, and the inequality on the right is a consequence of ℓ being nonnegative. □

Proof of Lemma 1.25. As mentioned above, we shall use in a crucial way the identity $\langle \ell, 1 \rangle = 1$, which is a particular case of (1.25).

Let (K_n) (resp. (L_n)) be an increasing sequence of compact sets in X (resp. Y), such that K_{n+1} is a neighborhood of K_n and $\bigcup K_n = X$ (resp. L_{n+1} is a neighborhood of L_n and $\bigcup L_n = Y$). We introduce an increasing sequence of cut-off functions $\varphi_n \in C_0(X)$ with $0 \leq \varphi_n \leq 1$ on X, $\varphi_n = 1$ on K_n and $\varphi_n = 0$ on K_{n+1}^c; and similarly, an increasing sequence of cut-off functions $\psi_n \in C_0(Y)$ with $0 \leq \psi_n \leq 1$ on Y, $\psi_n = 1$ on L_n and $\psi_n = 0$ on L_{n+1}^c. The existence of these sequences follows from Urysohn's lemma. In particular, we have

$$(1 - \varphi_{n+1})\psi_n + (1 - \psi_{n+1})\varphi_n \leq 1$$

on the whole of $X \times Y$, for all n; and

$$(1 - \varphi_{n+1})\psi_n + (1 - \psi_{n+1})\varphi_n \xrightarrow[n \to \infty]{} 0,$$

pointwise on $X \times Y$.

Let us introduce the quantity

$$A_n = \big\langle \ell,\, (1-\varphi_{n+1})\psi_n + (1-\psi_{n+1})\varphi_n \big\rangle. \tag{1.27}$$

On one hand, it is equal to

$$\int_{X\times Y} \big[(1-\varphi_{n+1})\psi_n + (1-\psi_{n+1})\varphi_n\big]\, d\pi(x,y) \\ + \big\langle R,\, (1-\varphi_{n+1})\psi_n + (1-\psi_{n+1})\varphi_n \big\rangle$$

$$\leq \int_{X\times Y} \big[(1-\varphi_{n+1})\psi_n + (1-\psi_{n+1})\varphi_n\big]\, d\pi(x,y) + \langle R, 1\rangle,$$

which converges as $n \to \infty$ to

$$\langle R, 1\rangle = \langle \ell, 1\rangle - \int_{X\times Y} d\pi = 1 - \int_{X\times Y} d\pi.$$

Thus,

$$\limsup_{n\to\infty} A_n \leq 1 - \pi[X\times Y]. \tag{1.28}$$

On the other hand, (1.27) can be rewritten as

$$\langle \ell, \varphi_n\rangle + \langle \ell, \psi_n\rangle - \langle \ell, \varphi_{n+1}\psi_n\rangle - \langle \ell, \varphi_n\psi_{n+1}\rangle.$$

By (1.25) and the fact that R is supported at infinity, this is also

$$\int_X \varphi_n\, d\mu + \int_Y \psi_n\, d\nu - \int_{X\times Y} \big(\varphi_{n+1}\psi_n + \varphi_n\psi_{n+1}\big)\, d\pi,$$

which converges as $n \to \infty$ to

$$1 + 1 - 2\int_{X\times Y} d\pi.$$

Thus,

$$\lim_{n\to\infty} A_n = 2 - 2\int_{X\times Y} d\pi. \tag{1.29}$$

From (1.28) and (1.29), it follows that

$$\pi[X\times Y] \geq 1.$$

In particular, $\langle R, 1\rangle = \langle \ell, 1\rangle - \langle \pi, 1\rangle \leq 0$, which implies that $\langle R, 1\rangle = 0$, so that in fact $R = 0$. This concludes the proof. □

With Lemma 1.25 in hand, the proof of Proposition 1.22 will be straightforward.

Proof of Proposition 1.22. Since μ and ν have σ-compact supports by Ulam's lemma, we may assume without loss of generality that X and Y are σ-compact.

We introduce again Θ and Ξ, defined as in Step 1 of the proof of Theorem 1.3, on $C_b(X \times Y)$. If we want to apply Theorem 1.9, we immediately run into a problem: we would like to take $z_0 = 0$, but Θ is plainly not continuous at 0 whenever $\inf c(x, y) = 0$ (which is usually the case, since it is natural to assume $c(x, x) = 0$ if $X = Y$). This difficulty is however easily remedied: it suffices to replace c by $c_\varepsilon = c + \varepsilon$, for some arbitrary $\varepsilon > 0$. Indeed, with this modified cost, $\|u\|_\infty \le \varepsilon \Rightarrow \Theta(u) = 0$, so the assumptions of Theorem 1.9 are fulfilled with $z_0 = 0$, and (1.9) holds.

Since (as one readily checks) adding ε to the cost function results in adding ε to both sides of (1.4), the result for the modified cost $c + \varepsilon$ will imply the result for the original cost c. In the sequel, we write c in place of c_ε. So we have the duality formula

$$\sup_{u \in C_b(X \times Y)} [-\Theta(u) - \Xi(u)] = \inf_{\ell \in C_b(X \times Y)^*} [\Theta^*(-\ell) + \Xi^*(\ell)]. \tag{1.30}$$

There is no difficulty in evaluating the left-hand side, and we concentrate on the right-hand side.

Let us first look at the Legendre transform of Θ. Let $\ell \in C_b(X \times Y)^*$; we have

$$\Theta^*(-\ell) = \sup_{u \ge -c} \langle -\ell, u \rangle = \sup_{u \le c} \langle \ell, u \rangle.. \tag{1.31}$$

Suppose that ℓ is not a nonnegative linear form; then, by definition there exists some $v \in C_b(X \times Y)$ such that $v \le 0$ but $\langle \ell, v \rangle > 0$. Then, for any $\lambda > 0$, we have $\lambda v \le c$, and by using λv as a trial function, we see that the supremum in (1.31) is $+\infty$, which will not contribute to the infimum in (1.30). We therefore only take into account the case when ℓ is nonnegative.

Let us now have a look at Ξ^*. It is easy to check again that

$$\Xi^*(\ell) = \begin{cases} 0 & \text{if } \Big[\forall (\varphi, \psi) \in C_b(X) \times C_b(Y), \\ & \qquad \langle \ell, \varphi + \psi \rangle = \int_X \varphi \, d\mu + \int_Y \psi \, d\nu \Big], \\ +\infty & \text{else,} \end{cases}$$

with the shorthand $(\varphi + \psi)(x, y) = \varphi(x) + \psi(y)$. In particular, when ℓ is nonnegative, by using Lemma 1.25 we see that $\Xi^*(\ell)$ is 0 if $\ell \in \Pi(\mu, \nu)$, and $+\infty$ otherwise. One can then conclude as in Step 1 of the proof of Theorem 1.3. □

1.4. Appendix: $\{0,1\}$-valued costs and Strassen's theorem

The Kantorovich duality takes a particular form when the cost function c only takes values 0 and 1, i.e. when it is of the form $1_C(x,y)$:

Theorem 1.27 (Kantorovich duality for $\{0,1\}$-valued costs). *Let X and Y be Polish spaces, $\mu \in P(X)$, $\nu \in P(Y)$, and let C be a nonempty open set in $X \times Y$. Then,*

$$\inf_{\pi\in\Pi(\mu,\nu)} \pi[C] = \sup\Big\{\mu[A] - \nu[A^C]; \quad A \subset X,\ A \text{ closed}\Big\},$$

where

$$A^C := \{y \in Y;\ \exists x \in A,\ (x,y) \notin C\}.$$

This theorem contains a measure-theoretical version of the **marriage lemma** , or pairing theorem; we shall come back to this at the end of the Appendix.

The following corollary is a famous and useful theorem due to Strassen. To state it, we use the standard notation $d(y,A) = \inf\{d(y,z);\ z \in A\}$.

Corollary 1.28 (Strassen's theorem). *Let X be a Polish space, $\mu,\nu \in P(X)$, and $\varepsilon \geq 0$. Then*

$$\inf_{\pi\in\Pi(\mu,\nu)} \pi\big[\{d(x,y) > \varepsilon\}\big] = \sup_{A \text{ closed}} \Big\{\mu[A] - \nu[A^\varepsilon]\Big\},$$

where $A^\varepsilon = \{y \in X;\ d(y,A) \leq \varepsilon\}$.

Note that the case $\varepsilon = 0$ reduces to (14). Dudley [**124**, Lecture 18] [**127**, Section 11.6] proves a very slight variation of Strassen's theorem without the assumption of X being complete; his argument is based on the marriage lemma. Another proof is given in Rachev and Rüschendorf [**225**]. It is in fact possible to prove Strassen's theorem by direct use of the Hahn-Banach theorem; so our goal here is not to provide the simplest proof, but only to show that this theorem can be seen as a particular case of the Kantorovich duality.

Remark 1.29. From Strassen's theorem follows the equality

$$\begin{aligned} \inf\{\varepsilon > 0;\ \text{for any closed } A,\ \mu[A] - \nu[A^\varepsilon] \leq \varepsilon\} \\ = \inf\{\varepsilon > 0;\ \inf \mathbf{P}[d(U,V) > \varepsilon] \leq \varepsilon\}, \end{aligned} \tag{1.32}$$

where the last infimum runs over all random variables U and V with respective laws μ and ν. The expression in (1.32) is a distance metrizing weak topology, called the **Prokhorov distance** (or sometimes Lévy distance).

Proof of Theorem 1.27. 1. Since C is open, the cost function $c(x,y) = 1_C(x,y)$ is lower semi-continuous on $X \times Y$, and can be approximated

pointwise by a nondecreasing sequence (c_k) of continuous functions, with $0 \le c_k \le c$. As we saw in the proof of Theorem 1.3,

$$\begin{aligned}\inf_{\pi\in\Pi(\mu,\nu)} \pi[C] &= \lim_{k\uparrow\infty} \inf_{\pi\in\Pi(\mu,\nu)} \int c_k\, d\pi \\ &= \lim_{k\uparrow\infty} \sup\left\{\int_X \varphi\, d\mu + \int_Y \psi\, d\nu;\quad (\varphi,\psi)\in\Phi_{c_k}\right\}.\end{aligned}$$

In view of Remark 1.13, for each k we may restrict the supremum to those pairs (φ,ψ) in Φ_{c_k} such that

$$0\le\varphi\le 1, \qquad -1\le\psi\le 0,$$

and φ, ψ are upper semi-continuous (as infima of continuous functions, see formula (1.18)) and satisfy the inequality $\varphi(x)+\psi(y)\le c_k(x,y)\le c(x,y)$ for *all* x,y. We deduce that

$$\inf_{\pi\in\Pi(\mu,\nu)} \pi[C] = \sup\left\{\int \varphi\, d\mu + \int \psi\, d\nu;\quad (\varphi,\psi)\in\widetilde{\Phi}_c\right\}, \tag{1.33}$$

where $\widetilde{\Phi}_c$ is the set of all pairs (φ,ψ) in $L^1(d\mu)\times L^1(d\nu)$ such that

$$\begin{cases}\varphi(x)+\psi(y)\le c(x,y)=1_C(x,y) & \text{for all } (x,y),\\ 0\le\varphi\le 1,\quad -1\le\psi\le 0, & \\ \varphi \text{ is upper semi-continuous.} & \end{cases} \tag{1.34}$$

Note that $\widetilde{\Phi}_c$ is a convex set.

2. We claim that each $(\varphi,\psi)\in\widetilde{\Phi}_c$ can be represented as a convex combination of pairs of the form $(1_A,-1_B)$, where A is closed, belonging to $\widetilde{\Phi}_c$ (i.e. $1_A(x)-1_B(y)\le 1_C(x,y)$ for all x,y). Let us postpone the proof of this claim for a while. Since the functional to maximize, $J(\varphi,\psi) = \int\varphi\, d\mu + \int\psi\, d\nu$, is linear, we deduce that for all $(\varphi,\psi)\in\widetilde{\Phi}_c$, there exists such a pair $(1_A,-1_B)$ with $J(1_A,-1_B)\ge J(\varphi,\psi)$. In particular, the value of the right-hand side in (1.33) is unchanged if one restricts the supremum to pairs of the form $(1_A,-1_B)$. Once this is proven, Theorem 1.27 will follow: indeed, $1_A-1_B\le 1_C$ implies that for all y,

$$1_B(y)\ge \sup_{x\in X}[1_A(x)-1_C(x,y)] = 1_{A^C}(y),$$

which means $A^C\subset B$, so

$$\mu[A]-\nu[B]\le\mu[A]-\nu[A^C].$$

3. We prove the claim made above, by using the **"layer cake representation"**: any measurable mapping $u: X\to[0,1]$ can be written as

$u = \int_0^1 1_{u\geq s}\, ds$ (check!). In particular, whenever $(\varphi, \psi) \in \widetilde{\Phi}_c$, one can write

$$(\varphi, \psi) = \int_0^1 \left(1_{\varphi\geq s}, -1_{\psi\leq -s}\right) ds.$$

Since φ is upper semi-continuous, the set $1_{\varphi\geq s}$ is closed, for all $s \in [0,1]$. So we only have to check that for all s,

$$1_{\varphi\geq s}(x) - 1_{\psi\leq -s}(y) \leq 1_{C(x,y)}.$$

The only nontrivial case is when $\varphi(x) \geq s$ and $\psi(y) > -s$; in this situation (x,y) should belong to C. But this case implies $\varphi(x)+\psi(y) > 0$, in particular

$$c(x,y) \geq \varphi(x) + \psi(y) > 0.$$

Since c takes values in $\{0,1\}$, we deduce that $c(x,y) = 1$, which indeed means $(x,y) \in C$. The proof is complete. □

Remark 1.30. In fact, the reasoning shows that for $\{0,1\}$-valued cost functions, in the Kantorovich duality one may restrict to pairs of c-concave functions of the form $(1_A, -1_{A^c})$.

Let us conclude this chapter by making the connection with the marriage lemma. (The idea of this connection came after a discussion with E. Bachmat.) And for a start, let us explain what is the marriage lemma. Think of $X = \{x_1, \ldots, x_N\}$ as a set of boys, and $Y = \{y_1, \ldots, y_N\}$ as a set of girls. They all have preferrences: each boy x would be happy to marry a girl from a certain subset of Y, and conversely each girl y would be happy to marry a boy from a certain subset of X. Considering only mutual inclinations, we may model this by a relation $\Gamma \subset X \times Y$: whenever $(x_i, y_j) \in \Gamma$ this means that x_i and y_j are ready to get married. The question is whether there is a way for everybody to be married in such a way that everybody is happy; in mathematical terms, this means finding a bipartite matching inside Γ. And the answer is that this is possible if and only if (a) for each subset $X' \subset X$ of cardinality $m \leq N$, there are at least m elements in Y which match some element in X'; (b) for each subset $Y' \subset Y$ of cardinality $m \leq N$, there are at least m elements in X which match some element in Y'. In other words, for each subset of boys/girls, one should find at least as many interested girls/boys. (In fact conditions (a) and (b) turn out to be equivalent, which is not obvious.)

The marriage lemma is of combinatorial nature, and is useful in various problems related to bipartite matching. It turns out that Theorem 1.27 implies a generalization of the marriage lemma:

Theorem 1.31 (Generalized Marriage Lemma). *Let X and Y be two Polish spaces; let μ, ν be probability measures on X, Y respectively; and let*

$\Gamma \subset X \times Y$ be closed. If $A \subset X$ is given, write

$$\Gamma_Y^{-1}(A) = \Big\{ y \in Y;\ \exists x \in A;\ (x,y) \in \Gamma \Big\}.$$

Symetrically, if $B \subset Y$ is given, write

$$\Gamma_X^{-1}(B) = \Big\{ x \in X;\ \exists y \in B;\ (x,y) \in \Gamma \Big\}.$$

Then there exists $\pi \in \Pi(\mu,\nu)$ concentrated on Γ, if and only if any one of the following equivalent properties holds true:

(a) for any closed set $A \subset X$, $\mu[A] \le \nu[\Gamma_Y^{-1}(A)]$;

(b) for any closed set $B \subset Y$, $\nu[B] \le \mu[\Gamma_X^{-1}(B)]$.

Proof of Theorem 1.31. First note that the existence of π implies both (a) and (b): for instance,

$$\mu[A] = \pi[A \times Y] = \pi[A \times \Gamma_Y^{-1}(A)] \le \pi[X \times \Gamma_Y^{-1}(A)] = \nu[\Gamma_Y^{-1}(A)].$$

Conversely, let us assume (a) and prove the existence of π (which will then imply (b) as well). Without loss of generality, we may assume $\Gamma \ne X \times Y$ (otherwise the conclusion is obvious) and define C as the complement of Γ:

$$(x,y) \in C \ \Leftrightarrow (x,y) \notin \Gamma.$$

Then from Theorem 1.27,

$$\inf \pi[C] = \sup_{A \text{ closed}} \Big\{ \mu[A] - \nu[\Gamma_Y^{-1}(A)] \Big\}.$$

The infimum on the left-hand side really is a minimum; and it is equivalent to say that π is concentrated on Γ, or that $\pi[C] = 0$. So there exists such a π if and only if

$$\sup_{A \text{ closed}} \Big\{ \mu[A] - \nu[\Gamma_Y^{-1}(A)] \Big\} \le 0.$$

But this is precisely condition (a), and thus the proof is complete. □

Chapter 2

Geometry of Optimal Transportation

We now turn to the basic problem of the identification of optimal transference plans. We shall give a prominent role to the case of a quadratic cost function in $\mathbb{R}^n$, because this is the case where the results are most simple, and most important for applications. The fundamental result is the following: when the probability measures μ and ν under consideration have finite moments of order 2, a transference plan is optimal *if and only if it is concentrated on the subdifferential of a convex function.* This is the Knott-Smith optimality criterion, rediscovered by Brenier. Moreover, under a very weak "regularity" condition (absolute continuity, for instance), one can prove that there exists a unique such transference plan. This result is often referred to as Brenier's theorem.

There are really two different classes of proofs for this theorem: those which are based on Kantorovich's duality principle (historically the first), and those which are not! In Section 2.1, two duality-based arguments are presented. The first one, adapted from Brenier, is simpler but requires more restrictive assumptions, while the second one, taken from Rachev and Rüschendorf, is more general. Whatever the proof, the key idea is that, under reasonable assumptions, in the optimal transport plan, mass flows along trajectories which are determined by the optimal price function from the Kantorovich dual problem.

An elementary but important variant of this duality strategy, which does not make explicit use of the Kantorovich duality theorem, will be presented next, along the lines proposed independently by Gangbo and by Caffarelli

(the latter following a suggestion by Varadhan). Some reminders about convex functions are included within the first section, as a digression.

In Section 2.2, we investigate the particular case of the real line. The corresponding results have been known for quite a while, and go back to the works of Hoeffding and Fréchet. It is possible to prove them in a much more elementary way, but we prefer to see them as corollaries of the general theorems proven in Section 2.1.

In Section 2.3, an alternative approach is sketched, still for the quadratic cost. It relies more directly on geometrical ideas, in particular the concept of *cyclical monotonicity*. Even though Knott and Smith on one hand, Brenier on the other, pointed out this line of reasoning, it was McCann who first took full advantage of it, in conjunction with a lemma by Aleksandrov.

As a remarkable by-product of this study, we shall discover that, whenever μ and ν are two absolutely continuous probability measures on $\mathbb{R}^n$, there is a ($d\mu$-almost everywhere) unique mapping of the form $x \longmapsto \nabla\varphi(x)$, with φ convex, transporting μ onto ν. In this perspective optimal transportation can be seen as an elegant way to construct this mapping.

The optimal transportation theorem for quadratic cost is so neat that it may look "too good to be true". Actually, this case is so peculiar that it conceals the general structure of the theorem. For this reason, it is important to take a look at more general results even if one is ultimately interested in the quadratic cost function only. In Section 2.4, we review such generalizations, considering cost functions of the form $|x-y|^p$, or defined in terms of the geodesic distance on a Riemannian manifold. The concept of abstract c-concavity, implicit in Chapter 1, will be studied briefly, and some connections with the Kantorovich duality will be discussed. Finally, in Section 2.5, we shall collect a few facts about c-concave functions for strictly convex costs.

In this chapter, only the main results of Chapter 1 will be used, and we shall allow for a little bit of repetition as far as the methods and arguments are concerned (for instance, the proof of existence of a minimizer for the Kantorovich problem will be recast). This makes it possible to go through the present chapter without having read Chapter 1.

2.1. A duality-based proof for the quadratic cost

Our final goal in this section is Theorem 2.12 on p. 68. Before arriving there, we shall go through a series of preliminaries. We shall use the same notation as in the Introduction and Chapter 1. We let $X = Y = \mathbb{R}^n$, and the cost function will be the square of the Euclidean norm. Actually, for elegance of proofs we shall define $c(x,y) = |x-y|^2/2$ instead of $|x-y|^2$; of course, this

factor 1/2 has no influence on the Kantorovich problem. So in this section

$$I[\pi] = \int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|x-y|^2}{2}\, d\pi(x,y). \tag{2.1}$$

Let μ, ν be two (Borel) probability measures on $\mathbb{R}^n$ with *finite second order moments*, i.e.

$$M_2 := \int_{\mathbb{R}^n} \frac{|x|^2}{2}\, d\mu(x) + \int_{\mathbb{R}^n} \frac{|y|^2}{2}\, d\nu(y) < +\infty. \tag{2.2}$$

This condition (which can be relaxed, see Section 2.3 below) ensures that the functional I is always finite on $\Pi(\mu,\nu)$. Indeed, whenever $\pi \in \Pi(\mu,\nu)$,

$$I[\pi] = \int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|x-y|^2}{2}\, d\pi(x,y) \leq \int (|x|^2 + |y|^2)\, d\pi(x,y) = 2M_2.$$

Recall from Chapter 1 the Kantorovich duality principle,

$$\inf_{\pi \in \Pi(\mu,\nu)} I[\pi] = \sup_{(\varphi,\psi)\in\Phi_c} J(\varphi,\psi). \tag{2.3}$$

This will be the starting point of our investigation.

2.1.1. The primal problem.

As our first step, we give a very rough answer to the primal Kantorovich minimization problem.

Proposition 2.1 (Existence of an optimal measure). *The minimization problem* $\inf\{I[\pi];\ \pi \in \Pi(\mu,\nu)\}$ *admits a minimizer.*

The proof has already been given in Chapter 1; we repeat it briefly to make this chapter as self-contained as possible. The argument only uses the lower semi-continuity of the cost function.

Proof of Proposition 2.1. First of all, as noticed before, $\Pi(\mu,\nu)$ is nonempty. The key point consists in noting that $\Pi(\mu,\nu)$ is compact for the weak topology of probability measures (the topology induced by $C_b(\mathbb{R}^n \times \mathbb{R}^n)$). Here this can be seen (why?) as a consequence of the inequality

$$\pi \in \Pi(\mu,\nu) \Longrightarrow \int_{\mathbb{R}^n \times \mathbb{R}^n} (|x|^2 + |y|^2)\, d\pi(x,y) \leq 2M_2 < +\infty;$$

but in fact this is a general property which does not even need finiteness of M_2 or the fact that $X = Y = \mathbb{R}^n$. Indeed, let $\delta > 0$ be given, and let $K \subset X$, $L \subset Y$ be such that

$$\mu[X \setminus K] \leq \delta, \qquad \nu[Y \setminus L] \leq \delta.$$

Whenever $\pi \in \Pi(\mu,\nu)$,

$$\pi[(X \times Y) \setminus (K \times L)] \leq \pi[X \times (Y \setminus L)] + \pi[(X \setminus K) \times L] \leq 2\delta.$$

This shows the tightness of the set $\Pi(\mu,\nu)$, and therefore its relative compactness with respect to the weak topology. Moreover, since the conditions which define $\Pi(\mu,\nu)$ are continuous with respect to the weak topology, we see that $\Pi(\mu,\nu)$ is weakly closed, hence compact.

This implies at once that there exists a minimizer for I. Indeed, let $(\pi_k)_{k\in\mathbb{N}}$ be a minimizing sequence; then it admits a cluster point $\pi_* \in \Pi(\mu,\nu)$. Write the cost function $c(x,y) = |x-y|^2/2$ as the supremum of a nondecreasing sequence $(c_\ell)_{\ell\in\mathbb{N}}$ of bounded continuous functions. By invoking successively the monotone convergence theorem, the fact that π_* is a cluster point, the inequality $c_\ell \le c$ and the minimizing property of (π_k), we obtain

$$\begin{aligned}\int c(x,y)\,d\pi_*(x,y) &= \lim_{\ell\to\infty}\int c_\ell(x,y)\,d\pi_*(x,y)\\ &\le \lim_{\ell\to\infty}\limsup_{k\to\infty}\int c_\ell(x,y)\,d\pi_k(x,y)\\ &\le \limsup_{k\to\infty}\int c(x,y)\,d\pi_k(x,y) = \inf I.\end{aligned}$$

So π_* is actually a minimizer of I. □

2.1.2. The dual problem. As our second step, we study the dual problem, whose particular structure will lead to a much more precise result.

The condition for (φ,ψ) to belong to Φ_c is

$$\varphi(x)+\psi(y) \le \frac{|x-y|^2}{2}, \tag{2.4}$$

holding true for $d\mu$-almost all x and $d\nu$-almost all y in $\mathbb{R}^n$. Let us take advantage of the particular form of the quadratic cost function, and expand the right-hand side of (2.4). We find, after rearranging terms,

$$x\cdot y \le \left[\frac{|x|^2}{2}-\varphi(x)\right]+\left[\frac{|y|^2}{2}-\psi(y)\right]. \tag{2.5}$$

This is strongly reminiscent of the theory of convex conjugate functions, and suggests that we should consider as new unknowns

$$\widetilde{\varphi}(x) = \frac{|x|^2}{2}-\varphi(x), \qquad \widetilde{\psi}(y) = \frac{|y|^2}{2}-\psi(y).$$

In the sequel, for notational convenience we shall forget about the $\widetilde{\ }$ symbol. Using (2.2), we see that

$$\inf_{\Pi(\mu,\nu)} I[\pi] = M_2 - \sup\left\{\int (x\cdot y)\,d\pi(x,y);\quad \pi\in\Pi(\mu,\nu)\right\}, \tag{2.6}$$

and

$$\sup_{\Phi_c} J = M_2 - \inf\left\{J(\varphi,\psi);\quad (\varphi,\psi)\in\widetilde{\Phi}\right\}, \tag{2.7}$$

where $\widetilde{\Phi}$ is the set of all pairs (φ,ψ) in $L^1(d\mu)\times L^1(d\nu)$ (with values in $\mathbb{R}\cup\{+\infty\}$) such that for almost all x, y,

$$x\cdot y\le\varphi(x)+\psi(y). \tag{2.8}$$

Of course, (2.3) becomes

$$\sup\left\{\int(x\cdot y)\,d\pi(x,y);\quad \pi\in\Pi(\mu,\nu)\right\}=\inf\left\{J(\varphi,\psi);\quad(\varphi,\psi)\in\widetilde{\Phi}\right\}. \tag{2.9}$$

Now we introduce the **double convexification trick** to improve admissible pairs in the dual problem (this strategy was already useful in Chapter 1). Let $(\varphi,\psi)\in\widetilde{\Phi}$. By (2.8), for $d\nu$-almost all $y\in Y$ we can write

$$\psi(y)\ge\sup_x\,[x\cdot y-\varphi(x)]=:\varphi^*(y). \tag{2.10}$$

Remark 2.2. In the above formula, the $\sup_x$ may be understood as an essential supremum with respect to μ. But if we do so, we shall enter measure-theoretical nightmares (because a property which for all y holds true for almost all x, does not necessarily hold true for almost all x, for all y!....) from which it is better to keep away. For this reason, we adopt a slightly different strategy, as in the proof of Theorem 1.3: since $(\varphi,\psi)\in\widetilde{\Phi}$, we know that there exist measurable sets N_X, N_Y, with $\mu[N_X]=0$, $\nu[N_Y]=0$, such that inequality (2.8) holds true for all $(x,y)\in N_X^c\times N_Y^c$. We redefine φ to be $+\infty$ on N_X, and ψ to be $+\infty$ on N_Y. The resulting pair (φ,ψ) still belongs to $\widetilde{\Phi}$, and the value of $J(\varphi,\psi)$ is unchanged, since the modifications only affected negligible sets. Then we can write (2.10) with $\sup_x$ understood as a true supremum.

This remark is anyway not so important if one recalls from Chapter 1 that the Kantorovich duality still holds if φ,ψ are required to be continuous, and equation (2.8) is required to hold true for *all* x,y.

As a consequence of (2.10),

$$J(\varphi,\psi)\ge J(\varphi,\varphi^*). \tag{2.11}$$

Next, for $d\mu$-almost all $x\in X$,

$$\varphi(x)\ge\sup_y\,[x\cdot y-\varphi^*(y)]=:\varphi^{**}(x),$$

and therefore

$$J(\varphi,\varphi^*)\ge J(\varphi^{**},\varphi^*). \tag{2.12}$$

From (2.11) and (2.12) we see that

$$\inf_{(\varphi,\psi)\in\widetilde{\Phi}} J(\varphi,\psi) \geq \inf_{\varphi\in L^1(d\mu)} J(\varphi^{**},\varphi^*). \tag{2.13}$$

If we admit that $(\varphi^{**},\varphi^*) \in L^1(d\mu)\times L^1(d\nu)$, then clearly $(\varphi^{**},\varphi^*) \in \widetilde{\Phi}$, and it will follow from (2.13) that the infimum of J on $\widetilde{\Phi}$ is unchanged if one restricts J to the very small subset of $\widetilde{\Phi}$ which is made of pairs (φ^{**},φ^*). But φ^*, φ^{**} are very particular functions. They are *convex lower semi-continuous functions*, because each of them is defined as the supremum of a family of linear functions.

This is an opportunity to recall some well-known facts about convex functions. Once this digression is over, we shall come back to our study of the dual problem. The reader who feels very comfortable with convex functions can skip to subsection 2.1.4.

2.1.3. Digression: Some facts from convex analysis. A standard reference for convex analysis is Rockafellar [**228**], see especially Part V and Chapter 12. We also recommend the short section devoted to convex functions in Evans and Gariepy [**136**], and mention the nice study of regularity by Alberti, Ambrosio and Cannarsa [**4, 5**]. Below, we collect several results from these references, sketching the proofs only when useful for the applications we have in view.

In the theory of regularity of convex functions in $\mathbb{R}^n$, a natural notion of smallness is being of Hausdorff dimension at most $n-1$. From now on, sets of Hausdorff dimension at most $n-1$ in $\mathbb{R}^n$ will be referred to as **small sets**. Of course these sets are of zero Lebesgue measure. The reader who is not comfortable with Hausdorff dimension may replace "small" by "Lebesgue-negligible" throughout the text; this should not affect his or her understanding.

0) Definitions. A proper convex function φ on $\mathbb{R}^n$ is a function φ : $\mathbb{R}^n \to \mathbb{R}\cup\{+\infty\}$, not identically $+\infty$, such that

$$\forall x,y\in\mathbb{R}^n,\ \forall t\in[0,1], \qquad \varphi(tx+(1-t)y) \leq t\varphi(x)+(1-t)\varphi(y). \tag{2.14}$$

It is **strictly convex** if equality in (2.14) implies $x=y$, or $t=0$, or $t=1$. One defines $\mathrm{Dom}(\varphi)$ as the (convex) set of points where φ is finite. It may be open, or closed, or neither. Because it is a convex set, its boundary is a small set.

1) Differentiability. One proves that a proper convex function φ is automatically continuous and locally Lipschitz on $\mathrm{Int}(\mathrm{Dom}(\varphi))$, the largest open set included in $\mathrm{Dom}(\varphi)$. As a consequence, by **Rademacher's theorem**, $\nabla\varphi$ is well-defined almost everywhere and locally bounded (a proof of

Rademacher's theorem can be found in [**136**, subsection 3.1.2]). Moreover, the set of points where $\nabla\varphi$ does not exist (in the usual sense) is a small set. This theorem is due to Anderson and Klee; see Alberti and Ambrosio [**4**] for a very simple proof.

The values of φ on the frontier $\partial(\mathrm{Dom}(\varphi))$ can usually be modified in a variety of ways, without harming the convexity property of φ. But if φ is assumed to be lower semi-continuous, then this determines the values of φ on the frontier: two convex lower semi-continuous functions φ, ψ such that $\mathrm{Int}(\mathrm{Dom}(\varphi)) = \mathrm{Int}(\mathrm{Dom}(\psi))$, and both coincide on that set, are equal [**228**, Corollary 7.3.4].

2) The graph lies above its tangent. For all points x where φ is differentiable, we have the crucial relation

$$\forall z \in \mathbb{R}^n, \qquad \varphi(z) \geq \varphi(x) + \nabla\varphi(x) \cdot (z - x), \tag{2.15}$$

which expresses the geometrical fact that the whole graph of φ lies above its tangent hyperplane at point x. In particular, $\nabla\varphi$ is monotone: whenever φ is differentiable at points x and z,

$$\big\langle \nabla\varphi(x) - \nabla\varphi(z), x - z \big\rangle \geq 0.$$

3) Subdifferentiability. To deal with differentiability in spite of the possible non-differentiability of a convex function, one introduces the notion of **subdifferential**, considering (2.15) as a definition of gradient in a generalized sense. The subdifferential $\partial\varphi$ of a convex function φ is a set-valued application; by definition,

$$y \in \partial\varphi(x) \iff \Big[\forall z \in \mathbb{R}^n, \quad \varphi(z) \geq \varphi(x) + \langle y, z - x\rangle\Big]. \tag{2.16}$$

In these notes, we shall identify the subdifferential mapping $\partial\varphi$ with its graph, denoted by $\mathrm{Graph}\,(\partial\varphi)$, i.e. we shall consider the subdifferential as a subset of $\mathbb{R}^n \times \mathbb{R}^n$.

By using the Hahn-Banach separation theorem, one can show that for *all* (not just almost all) $x \in \mathrm{Int}(\mathrm{Dom}(\varphi))$, the subdifferential $\partial\varphi(x)$ is nonempty. Moreover, *φ is differentiable at a point x if and only if $\partial\varphi(x)$ contains a single element, which is then $\nabla\varphi(x)$.*

If φ is lower semi-continuous, then the subdifferential mapping $\partial\varphi$ is always "continuous" on the whole of $\mathbb{R}^n$, in the sense that

$$\left.\begin{array}{r} x_k \longrightarrow x \\ \partial\varphi(x_k) \ni y_k \longrightarrow y \end{array}\right\} \Longrightarrow y \in \partial\varphi(x). \tag{2.17}$$

In order to get a better feeling of the meaning of the subdifferential, we mention two theorems from [**228**]:

- The subdifferential mapping generates the **normal cone** to the sublevel sets of φ (see Figure 2.1). This means that if $C_x = \{z \in \mathbb{R}^n;\ \varphi(z) \leq \varphi(x)\}$, then the directions of all the vectors in $\partial\varphi(x)$ generate the normal cone

$$N_x = \Big\{n_x \in \mathbb{R}^n;\ \forall z \in C_x,\ \langle n_x, z - x\rangle \leq 0\Big\},$$

provided that $\varphi(x) \neq \inf \varphi$. This result generalizes the usual properties of the gradients, which in the case of smooth functions are orthogonal to level sets. (When $\varphi(x) = \inf \varphi$ the result does not hold: for instance in case of a unique minimum, $\partial\varphi(x) = \{0\}$, while $C_x = \mathbb{R}^n$.)

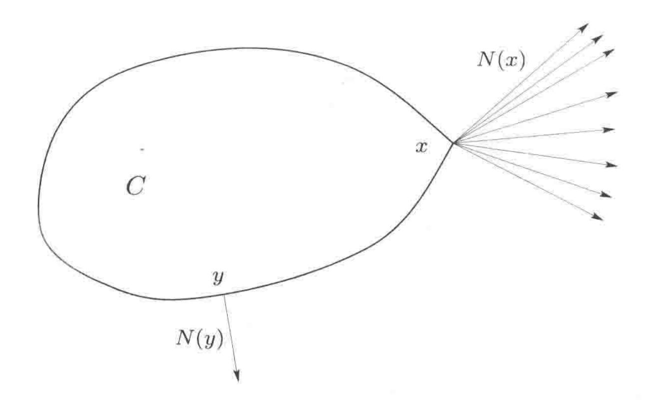


Figure 2.1. The normal cone is generated by the subdifferential.

- On Int(Dom(φ)),

$$\partial\varphi(x) = \overline{\mathrm{Conv}}\,(\lim_{x_k \to x} \nabla\varphi(x_k)).$$

In words, the subdifferential is the closed convex hull of (i.e. the smallest closed convex set containing) all the limits of $\nabla\varphi(z)$, as z approaches x (and φ is differentiable at z).

In dimension one, it is easy to obtain a complete description of subgradients (see Figure 2.2 and Section 2.2).

Exercise 2.3 ("Continuity of the gradient"). Let φ be a lower semicontinuous proper convex function, let x be a differentiability point of φ, and let $y = \nabla\varphi(x)$. Show that for any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\nabla\varphi(B_\delta(x)) \subset \partial\varphi(B_\delta(x)) \subset B_\varepsilon(y), \tag{2.18}$$

where $B_r(x_0)$ denotes the Euclidean ball of radius r centered at x_0.

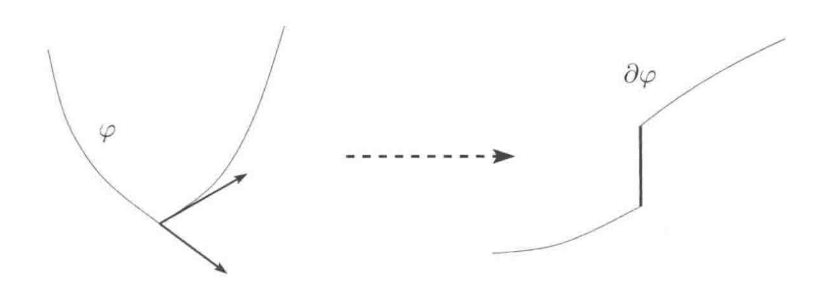


Figure 2.2. A one-dimensional subdifferential

4) Monotonicity. It is an immediate consequence of its definition that the subdifferential of a convex function φ is a *monotone mapping*: for all $y_1 \in \partial\varphi(x_1)$, $y_2 \in \partial\varphi(x_2)$,

$$\langle y_2 - y_1, x_2 - x_1 \rangle \geq 0.$$

This property can be used to prove that points of nondifferentiability form a small set.

5) Conjugate functions. For *any* proper (not identically $+\infty$) function $\varphi : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$, one can define its convex conjugate function, or Legendre transform, by

$$\varphi^*(y) = \sup_{x \in \mathbb{R}^n} (x \cdot y - \varphi(x)). \tag{2.19}$$

Then φ^* is a proper *convex* lower semi-continuous function. From the definition,

$$\forall x, y \in \mathbb{R}^n, \quad x \cdot y \leq \varphi(x) + \varphi^*(y). \tag{2.20}$$

The characterization of cases of equality in (2.20) will be crucial for us.

Proposition 2.4 (Characterization of subdifferential). *Let φ be a proper lower semi-continuous convex function on $\mathbb{R}^n$. Then, for all $x, y \in \mathbb{R}^n$,*

$$x \cdot y = \varphi(x) + \varphi^*(y) \Longleftrightarrow y \in \partial\varphi(x) \Longleftrightarrow x \in \partial\varphi^*(y).$$

Proof.

$$\begin{aligned}
x \cdot y = \varphi(x) + \varphi^*(y) &\overset{(2.20)}{\Longleftrightarrow} x \cdot y \geq \varphi(x) + \varphi^*(y) \\
&\overset{(2.19)}{\Longleftrightarrow} \forall z \in \mathbb{R}^n, \quad x \cdot y \geq \varphi(x) + y \cdot z - \varphi(z) \\
&\Longleftrightarrow \forall z \in \mathbb{R}^n, \quad \varphi(z) \geq \varphi(x) + \langle y, z - x \rangle \\
&\overset{(2.16)}{\Longleftrightarrow} y \in \partial\varphi(x).
\end{aligned}$$

By symmetry, the equivalence with $x \in \partial\varphi^*(y)$ is a consequence of point 7) below. □

6) Regularization. In convex analysis, what plays the role of the usual regularization by convolution is the **inf convolution**: whenever φ and ψ are two proper convex functions, define

$$(\varphi\Box\psi)(z) = \inf_{x+x'=z} [\varphi(x) + \psi(x')].$$

Obviously, if either φ or ψ is real-valued, so is $\varphi\Box\psi$.

In the same way as the usual convolution behaves remarkably under Fourier transform, the inf convolution behaves remarkably under Legendre transform:

$$(\varphi\Box\psi)^* = \varphi^* + \psi^*.$$

7) Duality and lower semi-continuity. The Legendre transform induces a duality on convex functions, or more precisely on lower semi-continuous proper convex functions. To illustrate this, we first state the important

Proposition 2.5 (Legendre duality for lower semi-continuous convex functions). *Let $\varphi : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a proper function. Then the following three properties are equivalent:*

(i) φ is convex lower semi-continuous;

(ii) $\varphi = \psi^*$ for some proper function ψ;

(iii) $\varphi^{**} = \varphi$.

Proof. It is obvious that (iii) implies (ii), which in turn implies (i). (Note: ψ has to be bounded below by some affine function, otherwise φ would be identically $+\infty$; the latter is excluded by assumption.) So the proof of Proposition 2.5 boils down to showing that if φ is convex lower semi-continuous, then $\varphi^{**} = \varphi$. We split the argument into three steps.

Step 1. From (2.20),

$$\varphi(x) \geq \sup_y [x \cdot y - \varphi^*(y)] = \varphi^{**}(x).$$

Step 2. Let $x \in \text{Int}(\text{Dom}(\varphi))$. Since $\partial\varphi(x) \neq \emptyset$, we can choose $y \in \partial\varphi(x)$. By Proposition 2.4, $\varphi(x) + \varphi^*(y) = x \cdot y$, so that

$$\varphi(x) \leq \sup_y [x \cdot y - \varphi^*(y)] = \varphi^{**}(x).$$

So φ and φ^{**} coincide on $\mathrm{Int}(\mathrm{Dom}(\varphi))$. In particular, $\varphi^{**} = \varphi$ if $\mathrm{Dom}(\varphi) = \mathbb{R}^n$.

Step 3. To treat the general case, we can regularize φ by inf convolution: let $\psi_\varepsilon(x) = |x|^2/(2\varepsilon)$, and

$$\varphi_\varepsilon = \varphi \Box \psi_\varepsilon.$$

Using the fact that φ is lower semi-continuous and bounded below by an affine function, it is easy (good exercise) to check that

$$\forall x \in \mathbb{R}^n, \qquad \varphi(x) = \lim_{\varepsilon \to 0} \varphi_\varepsilon(x).$$

Since $\mathrm{Dom}(\varphi_\varepsilon) = \mathbb{R}^n$, we know from Step 2 that $(\varphi_\varepsilon)^{**} = \varphi_\varepsilon$. On the other hand, $\varphi_\varepsilon \le \varphi$, so $\varphi_\varepsilon^* \ge \varphi^*$, so $\varphi_\varepsilon^{**} \le \varphi^{**}$. We conclude that for all $x \in \mathbb{R}^n$,

$$\varphi^{**}(x) \ge \liminf_{\varepsilon \to 0} \varphi_\varepsilon^{**}(x) = \liminf_{\varepsilon \to 0} \varphi_\varepsilon(x) = \varphi(x).$$

This, combined with Step 1, concludes the proof. □

Exercise 2.6. Show that the conclusion of Step 2 is a particular case of Theorem 1.9.

Remarks 2.7. (i) The geometrical fact underlying the proof is the following well-known statement: *a convex set is closed if and only if it is the intersection of all half-spaces containing it.* Here the convex set of interest is the epigraph of φ, which is closed if and only if φ is lower semi-continuous. As an exercise, the reader can show that this property implies statement (ii) of Proposition 2.5, and that (iii) in turn can be seen as an easy consequence of (ii) (**Hint:** See Exercise 2.35).

(ii) It is easy to construct convex functions φ which are not lower semi-continuous, in which case $\varphi^{**} \neq \varphi$. The discrepancy can however only occur on $\partial\mathrm{Dom}(\varphi)$.

The Legendre duality has important consequences at the level of differentiability. If φ is strictly convex in the neighborhood of some $x \in \mathbb{R}^n$, then φ^* is differentiable on $\partial\varphi(x)$, and $\nabla\varphi^*(y) = x$ for all $y \in \partial\varphi(x)$. If φ is differentiable and strictly convex, then so is φ^*, and $\nabla\varphi$ is one-to-one. By differentiation of (2.20), or by using the end of Proposition 2.4, one finds that

$$(\nabla\varphi)^{-1} = \nabla\varphi^*. \tag{2.21}$$

For this consult [**228**, Section 26]. If moreover φ is superlinear, i.e.

$$\lim_{|x| \to \infty} \frac{\varphi(x)}{|x|} = +\infty,$$

then $\nabla\varphi(\mathbb{R}^n) = \mathbb{R}^n$, in which case $\nabla\varphi$ is a bijection $\mathbb{R}^n \to \mathbb{R}^n$, and $\nabla\varphi^*$ is its inverse.

Remark 2.8. There is a duality correspondence between strict convexity of φ and smoothness of φ^*. If φ and φ^* are twice differentiable, this can be seen by the relation

$$D^2\varphi^*(\nabla\varphi(x)) = [D^2\varphi(x)]^{-1}, \tag{2.22}$$

which follows from differentiating (2.21).

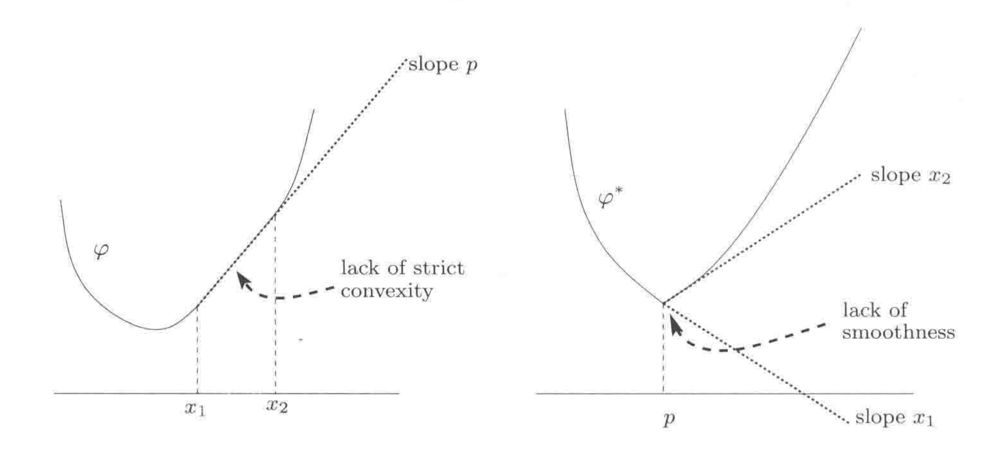


Figure 2.3. Duality between strict convexity and smoothness

8) Second differentiability. Actually, a convex function φ is automatically twice differentiable almost everywhere on $\mathrm{Int}(\mathrm{Dom}(\varphi))$. This is known as **Aleksandrov's theorem** [**6**]; see [**136**, pp. 241–245] for a modern proof. Here, twice differentiability is to be understood as follows: there exists a measurable matrix-valued function $D^2_A\varphi = [D^2\varphi]_{\mathrm{ac}}$, locally integrable and nonnegative in the sense of matrices, such that for almost all $x \in \mathrm{Int}(\mathrm{Dom}(\varphi))$,

$$\varphi(x+h) = \varphi(x) + \nabla\varphi(x)\cdot h + \langle [D^2\varphi]_{\mathrm{ac}}(x)\cdot h,\, h\rangle + o(|h|^2).$$

The notation $[D^2\varphi]_{\mathrm{ac}}$ comes from the fact that $[D^2\varphi]_{\mathrm{ac}}$ is really the absolutely continuous part of the distributional Hessian $D^2_{\mathcal{D}'}\varphi$. This distributional Hessian is the linear form defined on $\mathcal{D}(\Omega)$ by the identity

$$\forall \zeta \in \mathcal{D}(\Omega) \qquad \langle D^2_{\mathcal{D}'}\varphi, \zeta\rangle = \int_\Omega \varphi D^2\zeta.$$

Here $\Omega = \mathrm{Int}(\mathrm{Dom}(\varphi))$, and $\mathcal{D}(\Omega)$ stands for the vector space of C^∞ functions with compact support in Ω. It is easy to see that the distributional Hessian of a convex function is a nonnegative matrix-valued distribution, and this implies that it is a nonnegative matrix-valued *measure*. In particular, this matrix-valued measure is locally finite (each of its components has finite

total variation on any compact subset of Ω), and $\nabla\varphi$ lies in $BV_{\text{loc}}(\Omega;\mathbb{R}^n)$, the space of vector-valued functions whose partial derivatives have finite total variation, locally.

In this context, formula (2.22) is valid almost everywhere. More precisely: let x_0 be a point where φ is twice differentiable in the Aleksandrov sense; then $D^2\varphi(x_0)$ is invertible if and only if φ^* is twice differentiable at $\nabla\varphi(x_0)$ in the Aleksandrov sense, in which case formula (2.22) holds true. The proof of this fact can be found in [**203**, Proposition A.1].

9) Volume distortion. Our last remark concerns the interpretation of $D^2_A\varphi$ as a rate of local volume distortion. One can prove that whenever φ is twice differentiable at x_0 in the Aleksandrov sense, then

$$\frac{\big|\partial\varphi(B_r(x_0))\big|}{|B_r(x_0)|} \xrightarrow[r\to 0]{} \det D^2_A\varphi(x_0), \tag{2.23}$$

where $B_r(x_0)$ stands for the Euclidean ball of radius r centered at x_0. This is the content of [**203**, Proposition A.2].

One can also show that if $D^2_A\varphi(x_0)$ is invertible, then the ball $B_r(x_0)$ is not too much distorted under the action of $\partial\varphi$, in the following sense: there is a sequence $r_k \to 0$ such that $C_k \equiv \partial\varphi(B_{r_k}(x_0))$ contains a ball B_k whose volume is comparable, uniformly in k, to $|C_k|$ (meaning that $|B_k|/|C_k|$ is bounded below by a positive constant). And by convexity, this statement automatically implies that C_k is itself contained in a ball whose volume is comparable, uniformly in k, to $|C_k|$. A compact way to combine both statements is to write

$$B_k \subset C_k \subset B'_k,$$

where (B_k), (B'_k) are families of balls, and $|B_k|/|B'_k|$ is bounded below by a positive constant.

10) Variants. There are several variants of the concept of convexity. The most important ones are the following: a function φ is said to be λ-**uniformly convex** ($\lambda > 0$) if $D^2_{\mathcal{D}'}\varphi \geq \lambda I_n$ on $\mathbb{R}^n$, and **semi-convex** with constant $C > 0$ if $D^2_{\mathcal{D}'}\varphi \geq -CI_n$ on $\mathbb{R}^n$. These definitions make sense for functions which are locally integrable on the whole of $\mathbb{R}^n$. The following formulations, on the other hand, allow values in $\mathbb{R}\cup\{+\infty\}$: a function φ is said to be λ-uniformly convex (resp. semi-convex with a constant C) if $x \longmapsto \varphi(x) - \lambda|x|^2/2$ (resp. $x \longmapsto \varphi(x) + C|x|^2/2$) is convex.

It is easily checked that these notions respectively imply

$$\big\langle \nabla\varphi(x) - \nabla\varphi(y), x - y\big\rangle \geq \lambda|x-y|^2,$$

$$\big\langle \nabla\varphi(x) - \nabla\varphi(y), x - y\big\rangle \geq -C|x-y|^2.$$

Of course, by changing φ to $-\varphi$, one can define the concepts of concavity, strict concavity, uniform concavity, semi-concavity, and superdifferentiability of a concave function.

2.1.4. Back to the study of the dual Monge-Kantorovich problem.

Our goal now is to prove

Theorem 2.9 (Existence of an optimal pair of convex conjugate functions). *Let μ, ν be two probability measures on $\mathbb{R}^n$, with finite second order moments. Let $\widetilde{\Phi}$ be defined by (2.8). Then, there exists a pair (φ, φ^*) of lower semi-continuous proper conjugate convex functions on $\mathbb{R}^n$, such that*

$$\inf_{\widetilde{\Phi}} J = J(\varphi, \varphi^*).$$

Strategy of proof: Consider a minimizing sequence $(\varphi_k, \psi_k)_{k\in\mathbb{N}}$. By the double convexification trick, we may assume that (φ_k, ψ_k) are all pairs of conjugate convex functions (let us forget for the moment the need to prove their belonging to $L^1(d\mu) \times L^1(d\nu)$). Then, we would like to prove that, up to extraction of a subsequence, $\varphi_k \longrightarrow \varphi \in L^1(d\mu)$, $\psi_k \longrightarrow \psi \in L^1(d\nu)$, and

$$(\varphi, \psi) \in \widetilde{\Phi}, \quad J(\varphi, \psi) \le \liminf_{k\to\infty} J(\varphi_k, \psi_k).$$

So (φ, ψ) would be an optimal pair, and we would just have to double-convexify it, to get the desired result. □

However, *this convergence result cannot be true* in full generality, because of the invariance of the problem under addition of constants:

$$\forall a \in \mathbb{R}, \quad J(\varphi + a, \psi - a) = J(\varphi, \psi). \tag{2.24}$$

So, if (φ_k, ψ_k) is a minimizing sequence, converging in some sense to (φ, ψ), then also (say) $(\varphi_k + k, \psi_k - k)$ is minimizing, and we cannot hope that it will converge in any reasonable sense to a pair of L^1 functions. One first needs to ensure that (φ_k, ψ_k) "stay on a finite order of magnitude". The following important lemma allows us to do so, and provides L^1 estimates as well.

Lemma 2.10 (Double convexification lemma). *Let μ, ν be probability measures respectively supported in subsets X and Y of $\mathbb{R}^n$, satisfying*

$$M_2 \equiv \int_X \frac{|x|^2}{2}\, d\mu(x) + \int_Y \frac{|y|^2}{2}\, d\nu(y) < +\infty.$$

Whenever φ, ψ are measurable functions with values in $\mathbb{R} \cup \{+\infty\}$, introduce

$$\varphi^*(y) = \sup_{x\in X} [x \cdot y - \varphi(x)], \tag{2.25}$$

$$\psi^*(x) = \sup_{y\in Y} [x \cdot y - \psi(y)]. \tag{2.26}$$

Let $\widetilde{\Phi}$ be defined by (2.8)*, and let $(\varphi_k, \psi_k)_{k\in\mathbb{N}}$ be a minimizing sequence for J on $\widetilde{\Phi}$. Then,*

(i) One can modify (φ_k, ψ_k) on zero-measure sets (with respect to μ, ν) in such a way that inequality (2.8) *hold true for all $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$, without changing the values $J(\varphi_k, \psi_k)$.*

(ii) There exists a sequence of real numbers $(a_k)_{k\in\mathbb{N}}$ such that

$$(\overline{\varphi}_k, \overline{\psi}_k) = (\varphi_k^{**} - a_k, \varphi_k^* + a_k)$$

is still a minimizing sequence for J on $\widetilde{\Phi}$, and satisfies the lower bounds

$$\forall x \in X,\ \forall y \in Y, \qquad \overline{\varphi}_k(x) \geq -\frac{|x|^2}{2}, \quad \overline{\psi}_k(y) \geq -\frac{|y|^2}{2}, \tag{2.27}$$

together with the "upper bounds"

$$\liminf_{k\to\infty} \inf_{x\in X} \left(\overline{\varphi}_k(x) + \frac{|x|^2}{2} \right) \leq \inf_{\widetilde{\Phi}} J + M_2, \tag{2.28}$$

$$\liminf_{k\to\infty} \inf_{y\in Y} \left(\overline{\psi}_k(y) + \frac{|y|^2}{2} \right) \leq \inf_{\widetilde{\Phi}} J + M_2. \tag{2.29}$$

(iii) With the choice $X = Y = \mathbb{R}^n$, the $*$ operation coincides with the usual Legendre transform, and

$$\inf_{\widetilde{\Phi}} J = \inf_{\varphi\in L^1(d\mu)} J(\varphi^{**}, \varphi^*).$$

So the infimum of J on $\widetilde{\Phi}$ does not change upon restricting J to the narrow set of $\widetilde{\Phi}$ made of pairs of conjugate proper convex functions.

Remarks 2.11. (i) There is nothing special about the choice of $\mathbb{R}^n$ here. The same lemma holds true if X is an arbitrary Banach space E, and Y is its topological dual E^*, provided that the scalar product of $\mathbb{R}^n$ is replaced by the duality bracket between X and Y.

(ii) Without loss of generality, one may assume $X = Y = \mathbb{R}^n$. But sometimes it is convenient to use a different choice; see for instance the first proof of Theorem 2.12. Note that if X, Y do not coincide with the whole of $\mathbb{R}^n$, then φ^{**} and φ^*, as defined in Lemma 2.10, are not necessarily convex conjugate.

(iii) The strategy of proof below can be used for a more general Monge-Kantorovich problem, not necessarily set in a Banach space, under the assumption that the cost function c satisfies a bound like $c(x,y) \leq c_X(x) + c_Y(y)$, with $c_X \in L^1(d\mu)$, $c_Y \in L^1(d\nu)$. We already used similar tricks in the proof of Theorem 1.3.

Proof of Lemma 2.10. We already gave the proof of (i) in Remark 2.2, and (iii) follows from (ii) since $(\varphi_k^{**}-a_k, \varphi_k^*+a_k) = ((\varphi_k-a_k)^{**}, (\varphi_k-a_k)^*)$ is minimizing. So we just need to prove (ii).

1. Let (φ_k, ψ_k) be an arbitrary minimizing sequence for J. By invoking part (i) of the lemma, we may assume that for all x and y, $\varphi_k(x) \geq x \cdot y - \psi_k(y)$. Since ψ_k is not identically $+\infty$, we deduce that φ_k is bounded below (for all x) by some affine function, say $x \cdot y_0 + b_0$, for some $y_0 \in Y$. It follows that

$$\varphi_k^*(y_0) = \sup_{x \in X} [x \cdot y_0 - \varphi_k(x)] \leq -b_0.$$

In particular, φ_k^* is not identically $+\infty$.

Similarly, φ_k^* is bounded below by an affine function. This implies that

$$a_k \equiv \inf_{y \in Y} \left(\varphi_k^*(y) + \frac{|y|^2}{2} \right)$$

is finite. We choose

$$(\overline{\varphi}_k, \overline{\psi}_k) = \left(\varphi_k^{**} + a_k, \varphi_k^* - a_k \right).$$

Note that $\overline{\varphi}_k = (\overline{\psi}_k)^*$. By construction

$$\inf_{y \in Y} \left(\overline{\psi}_k(y) + \frac{|y|^2}{2} \right) = 0.$$

Moreover,

$$\overline{\varphi}_k(x) + \frac{|x|^2}{2} = (\overline{\psi}_k)^*(x) + \frac{|x|^2}{2} = \sup_{y \in Y} \left[x \cdot y - \overline{\psi}_k(y) + \frac{|x|^2}{2} \right]$$

$$\geq \sup_{y \in Y} \left[-\frac{|y|^2}{2} - \overline{\psi}_k(y) \right] = -\inf_{y \in Y} \left[\frac{|y|^2}{2} + \overline{\psi}_k(y) \right] = 0.$$

Thus the pair $(\overline{\varphi}_k, \overline{\psi}_k)$ satisfies (2.27).

2. We now check the integrability of $\overline{\varphi}_k$ and $\overline{\psi}_k$. By (2.24) and the reasoning in subsection 2.1.2,

$$J(\overline{\varphi}_k, \overline{\psi}_k) = J(\varphi_k^{**}, \varphi_k^*) \leq J(\varphi_k, \psi_k) < +\infty. \tag{2.30}$$

In particular,

$$\int_X \left(\frac{|x|^2}{2} + \overline{\varphi}_k(x) \right) d\mu(x) + \int_Y \left(\frac{|y|^2}{2} + \overline{\psi}_k(y) \right) d\nu(y) < +\infty.$$

Since both integrands are nonnegative, we deduce that they are integrable, and from the assumptions on μ, ν we finally deduce that $\overline{\varphi}_k \in L^1(d\mu)$, $\overline{\psi}_k \in L^1(d\nu)$. Then, because of their definition, $(\overline{\varphi}_k, \overline{\psi}_k) \in \widetilde{\Phi}$, and in view of the first inequality in (2.30), they constitute a minimizing sequence.

3. To complete the proof of the lemma, it only remains to check (2.28). For this we just have to note that

$$J(\overline{\varphi}_k, \overline{\psi}_k) + M_2 = \int_X \left(\overline{\varphi}_k(x) + \frac{|x|^2}{2} \right) d\mu(x) + \int_Y \left(\overline{\psi}_k(y) + \frac{|y|^2}{2} \right) d\nu(y)$$
$$\geq \inf_{x \in X} \left(\overline{\varphi}_k(x) + \frac{|x|^2}{2} \right) + \inf_{y \in Y} \left(\overline{\psi}_k(y) + \frac{|y|^2}{2} \right).$$

Since both expressions inside parentheses are nonnegative, it follows, for instance, that

$$\liminf_{k \to \infty} \inf_{x \in X} \left(\overline{\varphi}_k(x) + \frac{|x|^2}{2} \right) \leq \limsup_{k \to \infty} J(\overline{\varphi}_k, \overline{\psi}_k) + M_2 = \inf_{\widetilde{\Phi}} J + M_2.$$

This concludes the proof. □

With the help of this lemma, we are in a position to prove Theorem 2.9. We shall actually give two proofs. The first one is very simple but limited to the case when μ, ν are supported in bounded subsets of $\mathbb{R}^n$. The second one, taken from Rachev and Rüschendorf [**225**], holds in a much more general setting, since it will allow μ and ν to be supported in the whole of $\mathbb{R}^n$, and will also allow the replacement of $\mathbb{R}^n$ by an arbitrary Banach space.

First proof of Theorem 2.9. Assume that μ, ν are supported in compact sets X, Y of $\mathbb{R}^n$. Let (φ_k, ψ_k) be a minimizing sequence for J on $\widetilde{\Phi}$. Let $(\overline{\varphi}_k, \overline{\psi}_k)$ be a minimizing sequence constructed by Lemma 2.10. From the formula

$$\overline{\psi}_k(y) = \sup_{x \in X} [x \cdot y - \varphi_k(x) - a_k]$$

we deduce the Lipschitz bound

$$\|\overline{\psi}_k\|_{\mathrm{Lip}(Y)} \leq \sup_{x \in X} \|x\|.$$

Similarly,

$$\|\overline{\varphi}_k\|_{\mathrm{Lip}(X)} \leq \sup_{y \in Y} \|y\|.$$

Therefore $\overline{\varphi}_k$, $\overline{\psi}_k$ are uniformly Lipschitz.

On the other hand, from Lemma 2.10 we also know that for k large enough there exists $x_k \in X$, $y_k \in Y$ such that

$$-\sup_{x \in X} \frac{|x|^2}{2} \leq \overline{\varphi}_k(x_k) \leq \sup_{x \in X} \frac{|x|^2}{2} + \inf J + M_2 + 1,$$
$$-\sup_{y \in Y} \frac{|y|^2}{2} \leq \overline{\psi}_k(y_k) \leq \sup_{y \in Y} \frac{|y|^2}{2} + \inf J + M_2 + 1.$$

Combining this with the uniform Lipschitz bounds on $\overline{\varphi}_k$, $\overline{\psi}_k$ and the boundedness of X, Y, we see that $\overline{\varphi}_k$, $\overline{\psi}_k$ are uniformly bounded.

Since $\overline{\varphi}_k$, $\overline{\psi}_k$ are uniformly bounded and satisfy uniform Lipschitz estimates, we can invoke Ascoli's theorem to deduce that, up to extraction of a subsequence, they converge uniformly in $C_b(X)$, $C_b(Y)$ respectively, to continuous limits $\overline{\varphi}$, $\overline{\psi}$ respectively. By convergence in supremum norm, $J(\overline{\varphi}, \overline{\psi}) = \lim_{k\to\infty} J(\overline{\varphi_k}, \overline{\psi_k})$, so $(\overline{\varphi}, \overline{\psi})$ is an optimal pair (a priori defined on $X \times Y$ only). We can now extend $\overline{\varphi}$, $\overline{\psi}$ outside X, Y by $+\infty$ and double-convexify them in the usual sense (i.e., with the choice $X = Y = \mathbb{R}^n$ in Lemma 2.10) to find a pair of convex conjugate functions satisfying all the desired properties. □

Second proof of Theorem 2.9. The idea of this proof is to trade strong compactness for weak compactness plus monotonicity. We divide it into five steps.

Step 1. Replace the function $x \cdot y$ by a nonnegative function; for this add some well-chosen integrable functions to both sides of the inequality defining $\widetilde{\Phi}$. For instance,

$$(\varphi, \psi) \in \widetilde{\Phi} \Longleftrightarrow \left[\varphi(x) + \frac{|x|^2}{2}\right] + \left[\psi(y) + \frac{|y|^2}{2}\right] \geq \frac{|x|^2}{2} + \frac{|y|^2}{2} + x \cdot y = \frac{|x+y|^2}{2}.$$

Let $(\varphi_k, \psi_k)_{k\in\mathbb{N}}$ be a minimizing sequence for J. In view of Lemma 2.10, we can assume that

$$0 \leq \varphi_k(x) + \frac{|x|^2}{2}, \quad 0 \leq \psi_k(y) + \frac{|y|^2}{2},$$
$$\left[\varphi_k(x) + \frac{|x|^2}{2}\right] + \left[\psi_k(y) + \frac{|y|^2}{2}\right] \geq \frac{|x+y|^2}{2} \geq 0.$$

The important point here is that everything is nonnegative!

Step 2. Even if $\varphi_k(x) + |x|^2/2$ is bounded in $L^1(d\mu)$, this is not sufficient to ensure weak L^1 compactness: one would need some additional equi-integrability estimate. This is why we shall go through a truncation procedure. So, for each $\ell \in \mathbb{N}$, define $\varphi_k^{(\ell)}$, $\psi_k^{(\ell)}$ by truncation:

$$\varphi_k^{(\ell)}(x) + \frac{|x|^2}{2} = \min\left(\varphi_k(x) + \frac{|x|^2}{2}, \ell\right),$$
$$\psi_k^{(\ell)}(y) + \frac{|y|^2}{2} = \min\left(\psi_k(y) + \frac{|y|^2}{2}, \ell\right).$$

It is easy to check the following properties:

$$\begin{cases} 0 \leq \varphi_k^{(\ell)}(x) + \dfrac{|x|^2}{2} \leq \ell, \\ \\ 0 \leq \psi_k^{(\ell)}(y) + \dfrac{|y|^2}{2} \leq \ell, \end{cases} \tag{2.31}$$

$$\left\{\begin{array}{l}\varphi_k^{(1)} \le \varphi_k^{(2)} \le \ldots \le \varphi_k^{(\ell)} \le \ldots, \\ \\ \psi_k^{(1)} \le \psi_k^{(2)} \le \ldots \le \psi_k^{(\ell)} \le \ldots,\end{array}\right. \tag{2.32}$$

$$J(\varphi_k^{(\ell)}, \psi_k^{(\ell)}) \le J(\varphi_k, \psi_k), \tag{2.33}$$

$$\varphi_k^{(\ell)}(x) + \psi_k^{(\ell)}(y) \ge \min\left[\frac{|x+y|^2}{2}, \ell\right] - \left(\frac{|x|^2}{2} + \frac{|y|^2}{2}\right). \tag{2.34}$$

Step 3. By (2.31), we know that for each ℓ,

$$\varphi_k^{(\ell)}(x) = -\frac{|x|^2}{2} + O(\ell).$$

Since $-|x|^2/2$ is a fixed function in $L^1(d\mu)$, we deduce that $(\varphi_k^{(\ell)})_{k\in\mathbb{N}}$ defines a weakly compact sequence in $L^1(d\mu)$. Thus, up to extraction of a subsequence,

$$\varphi_k^{(\ell)} \longrightarrow \varphi^{(\ell)}, \quad \text{weakly in } L^1(d\mu), \text{ as } k \to \infty.$$

Similarly, for each ℓ the sequence $(\psi_k^{(\ell)})_{k\in\mathbb{N}}$ converges weakly in $L^1(d\nu)$, up to extraction of a subsequence, to some $\psi^{(\ell)} \in L^1(d\nu)$. By a *diagonal extraction*, we can extract a subsequence of $k \in \mathbb{N}$ for which the convergence holds for all ℓ. Then, since weak convergence preserves the ordering, we deduce from (2.32), (2.33), (2.34) that

$$\left\{\begin{array}{l}\varphi^{(1)} \le \varphi^{(2)} \le \ldots \le \varphi^{(\ell)} \le \ldots, \\ \psi^{(1)} \le \psi^{(2)} \le \ldots \le \psi^{(\ell)} \le \ldots,\end{array}\right. \tag{2.35}$$

$$J(\varphi^{(\ell)}, \psi^{(\ell)}) = \lim_{k\to\infty} J(\varphi_k^{(\ell)}, \psi_k^{(\ell)}) \le \liminf_{k\to\infty} J(\varphi_k, \psi_k) = \inf_{\widetilde{\Phi}} J, \tag{2.36}$$

$$\varphi^{(\ell)}(x) + \psi^{(\ell)}(y) \ge \min\left[\frac{|x+y|^2}{2}, \ell\right] - \left(\frac{|x|^2}{2} + \frac{|y|^2}{2}\right). \tag{2.37}$$

Step 4. The sequences $(\varphi^{(\ell)})$ and $(\psi^{(\ell)})$ are bounded in L^1, nondecreasing, and bounded below by a fixed L^1 function. Thus we can apply the

monotone convergence theorem, and deduce the existence of L^1 limits φ and ψ, defined almost everywhere:

$$\varphi = \lim_{\ell\to\infty} \varphi^{(\ell)}, \qquad \psi = \lim_{\ell\to\infty} \psi^{(\ell)},$$

satisfying

$$J(\varphi,\psi) = \lim_{\ell\to\infty} J(\varphi^{(\ell)},\psi^{(\ell)}) \leq \inf_{\widetilde{\Phi}} J.$$

By passing to the limit in (2.37), we see that $(\varphi,\psi) \in \widetilde{\Phi}$. So it is an optimizing pair.

Step 5. We double-convexify (φ,ψ) to get an optimizing pair made of conjugate proper convex functions. □

2.1.5. Optimal transportation theorem. At last, we are ready to prove our final goal in this section:

Theorem 2.12 (Optimal transportation theorem for quadratic cost). *Let μ, ν be probability measures on $\mathbb{R}^n$, with finite second order moments, in the sense of (2.2). We consider the Monge-Kantorovich transportation problem associated with a quadratic cost function $c(x,y) = |x-y|^2$. Then,*

(i) **(Knott-Smith optimality criterion)** *$\pi \in \Pi(\mu,\nu)$ is optimal if and only if there exists a convex lower semi-continuous function φ such that*

$$\operatorname{Supp}(\pi) \subset \operatorname{Graph}(\partial\varphi), \tag{2.38}$$

or equivalently:

$$\text{for } d\pi\text{-almost all } (x,y), \quad y \in \partial\varphi(x). \tag{2.39}$$

Moreover, in that case, the pair (φ,φ^*) has to be a minimizer in the problem

$$\inf\left\{\int_{\mathbb{R}^n} \varphi\, d\mu + \int_{\mathbb{R}^n} \psi\, d\nu; \qquad \forall (x,y), \quad x\cdot y \leq \varphi(x) + \psi(y)\right\}.$$

(ii) **(Brenier's theorem)** *If μ does not give mass to small sets, then there is a unique optimal π, which is*

$$d\pi(x,y) = d\mu(x)\,\delta[y = \nabla\varphi(x)], \tag{2.40}$$

or equivalently,

$$\pi = (\mathrm{Id} \times \nabla\varphi)\#\mu, \tag{2.41}$$

where $\nabla\varphi$ is the unique (i.e. uniquely determined $d\mu$-almost everywhere) gradient of a convex function which pushes μ forward to ν: $\nabla\varphi\#\mu = \nu$. Moreover,

$$\operatorname{Supp}(\nu) = \overline{\nabla\varphi(\operatorname{Supp}(\mu))}.$$

(iii) As a corollary, under the assumption of (ii), $\nabla\varphi$ is the unique solution to the Monge tranportation problem:

$$\int_{\mathbb{R}^n} |x - \nabla\varphi(x)|^2 \, d\mu(x) = \inf_{T\#\mu=\nu} \int_{\mathbb{R}^n} |x - T(x)|^2 \, d\mu(x),$$

or equivalently,

$$\int_{\mathbb{R}^n} x \cdot \nabla\varphi(x) \, d\mu(x) = \sup_{T\#\mu=\nu} \int_{\mathbb{R}^n} x \cdot T(x) \, d\mu(x).$$

(iv) Finally, if ν also does not give mass to small sets, then, for $d\mu$-almost all x and $d\nu$-almost all y,

$$\nabla\varphi^* \circ \nabla\varphi(x) = x, \qquad \nabla\varphi \circ \nabla\varphi^*(y) = y,$$

and $\nabla\varphi^*$ is the ($d\nu$-almost everywhere) unique gradient of a convex function which pushes ν forward to μ, and also the solution of the Monge problem for transporting ν onto μ with a quadratic cost function.

When the assumptions of (ii) hold true, we shall refer to the mapping $\nabla\varphi$ (uniquely defined $d\mu$-almost everywhere) as the **Brenier map** pushing μ forward to ν.

Remarks 2.13. (i) It is easy to see that there is not, in general, uniqueness for solutions of the Kantorovich problem: think about the example where μ, ν are probability measures in $\mathbb{R}^2$ concentrated on $\{(0,0),(1,1)\}$ and $\{(1,0),(0,1)\}$ respectively. Also note that there may be no measurable T such that $T\#\mu = \nu$. Thus it is natural to enforce some "regularity" assumption in (ii). The assumption that μ gives no mass to small sets is in some sense optimal, as shown by Exercise 2.14 below.

(ii) It makes sense to write $\nabla\varphi\#\mu$, even if $\nabla\varphi$ is not defined everywhere: by definition, $(\nabla\varphi)^{-1}(A)$ is the set of x's such that $\nabla\varphi(x)$ is well-defined and lies in A. Anyway, in the present context, this subtletly is of no importance, because μ will be concentrated on the differentiability set of φ. Again, the assumption that μ gives no mass to small sets is crucial for that.

(iii) To avoid confusion, note that the pair (φ, φ^*) does not solve the dual Monge-Kantorovich problem for quadratic cost: it is the pair

$$\left(\frac{|x|^2}{2} - \varphi(x), \; \frac{|y|^2}{2} - \varphi^*(y) \right)$$

which does so.

(iv) This theorem may look like nonsense to the reader with a geometrically oriented mind, because $\nabla\varphi$ is a tangent map. It might help to think of $\nabla\varphi(x)$

as

$$x + (\nabla\varphi(x) - x) = \exp_x \left[\nabla\Big|_x \left(\varphi - \frac{|\cdot|^2}{2} \right) \right],$$

where exp is the exponential map on the tangent bundle $T\mathbb{R}^n$, $\exp_x(z) = x + z$. See Theorem 2.47 below for more.

Proof of Theorem 2.12. Using the quadratic nature of the cost function, we reduce the Monge-Kantorovich minimization problem and its dual formulation to the problems (2.6)–(2.8) as before.

0. The equivalence between (2.40) and (2.41) has been seen in the Introduction: if $\pi \in \Pi(\mu, \nu)$ and $T\#\mu = \nu$, then

$$[d\pi\text{-almost everywhere}, y = T(x)] \Longleftrightarrow [\pi = (\mathrm{Id} \times T)\#\mu].$$

Recall from subsection 2.1.3 that Graph $(\partial\varphi)$ is closed, so it is really the whole support of π which lies in this graph.

1. By Proposition 2.1, there exists an optimal transference plan π. By Theorem 2.9, there exists a pair of convex conjugate functions (φ, φ^*), optimal for the dual problem. Then, by the Kantorovich duality of Chapter 1 (in the form of equation (2.9)) and the definition of $\Pi(\mu, \nu)$ (identity (2)),

$$\begin{aligned} \int_{\mathbb{R}^n \times \mathbb{R}^n} (x \cdot y)\, d\pi(x, y) &= \int_{\mathbb{R}^n} \varphi\, d\mu + \int_{\mathbb{R}^n} \varphi^*\, d\nu \\ &= \int_{\mathbb{R}^n \times \mathbb{R}^n} [\varphi(x) + \varphi^*(y)]\, d\pi(x, y). \end{aligned}$$

Equivalently,

$$\int_{\mathbb{R}^n \times \mathbb{R}^n} \big[\varphi(x) + \varphi^*(y) - x \cdot y\big]\, d\pi(x, y) = 0.$$

But the function inside square brackets is nonnegative by (2.20), so it has to vanish for $d\pi$-almost all (x, y). By Proposition 2.4, this entails (2.39).

2. Conversely, let $\pi \in \Pi(\mu, \nu)$ satisfy (2.39). Then, by the same arguments,

$$\int_{\mathbb{R}^n \times \mathbb{R}^n} (x \cdot y)\, d\pi(x, y) = \int_{\mathbb{R}^n} \varphi\, d\mu + \int_{\mathbb{R}^n} \varphi^*\, d\nu.$$

So π and (φ, φ^*) are optimal in both sides of (2.9). This concludes the proof of (i).

3. Now, let us assume that μ does not give mass to small sets, and let φ be as above. Since φ lies in $L^1(d\mu)$, it is $d\mu$-almost everywhere finite: $\mu[\mathrm{Dom}(\varphi)] = 1$. On the other hand, the border $\partial\mathrm{Dom}(\varphi)$ of the convex set $\mathrm{Dom}(\varphi)$ is a small set; so, $\mu[\mathrm{Int}(\mathrm{Dom}(\varphi))] = 1$. Now, on $\mathrm{Int}(\mathrm{Dom}(\varphi))$, the set of nondifferentiability of φ is a small set. On the whole, $d\mu$-almost every point of X is a differentiability point for φ. So, for $d\mu$-almost all x, the

subdifferential of φ at the point x is $\{\nabla\varphi(x)\}$. Recalling that a statement true for $d\mu$-almost all x is also true for $d\pi$-almost all (x,y), we obtain that $y = \nabla\varphi(x)$ for $d\pi$-almost all (x,y).

4. So far we have proven that any optimal transference plan takes the form $(\mathrm{Id} \times \nabla\varphi)\#\mu$, for some convex φ such that $\nabla\varphi\#\mu = \nu$, and we have shown that there is at least one such transference plan. We shall now prove uniqueness. Let $\overline{\varphi}$ be another convex function such that $\nabla\overline{\varphi}\#\mu = \nu$. We wish to prove that $\nabla\varphi = \nabla\overline{\varphi}$ up to a μ-negligible set. By (i), $(\mathrm{Id} \times \nabla\overline{\varphi})\#\mu$ is an optimal transference plan, and accordingly, $(\overline{\varphi}, \overline{\varphi}^*)$ is an optimal pair for the dual problem, just like (φ, φ^*). Therefore,

$$\int_{\mathbb{R}^n} \overline{\varphi}\, d\mu + \int_{\mathbb{R}^n} \overline{\varphi}^*\, d\nu = \int_{\mathbb{R}^n} \varphi\, d\mu + \int_{\mathbb{R}^n} \varphi^*\, d\nu. \tag{2.42}$$

Let π be the optimal transference plan *associated to* φ. From (2.42) we have

$$\begin{aligned}\int_{\mathbb{R}^n\times\mathbb{R}^n} [\overline{\varphi}(x) + \overline{\varphi}^*(y)]\, d\pi(x,y) &= \int_{\mathbb{R}^n\times\mathbb{R}^n} [\varphi(x) + \varphi^*(y)]\, d\pi(x,y) \\ &= \int_{\mathbb{R}^n\times\mathbb{R}^n} (x\cdot y)\, d\pi(x,y).\end{aligned}$$

Since $\pi = (\mathrm{Id} \times \nabla\varphi)\#\mu$, this can be rewritten as

$$\int_{\mathbb{R}^n} \left[\overline{\varphi}(x) + \overline{\varphi}^*(\nabla\varphi(x))\right] d\mu(x) = \int_{\mathbb{R}^n} x\cdot\nabla\varphi(x)\, d\mu(x).$$

Thus,

$$\int_{\mathbb{R}^n} \left[\overline{\varphi}(x) + \overline{\varphi}^*(\nabla\varphi(x)) - x\cdot\nabla\varphi(x)\right] d\mu(x) = 0.$$

Since the integrand is nonnegative, it has to vanish $d\mu$-almost everywhere. Using (2.4) again, we conclude that

$$\nabla\varphi(x) \in \partial\overline{\varphi}(x) \qquad \text{for } d\mu\text{-almost every } x.$$

Since $\overline{\varphi}$ is differentiable μ-almost everywhere, we end up with

$$\nabla\varphi(x) = \nabla\overline{\varphi}(x), \qquad \text{for } d\mu\text{-almost every } x,$$

which ends the proof of (ii). **Note carefully** that we have shown not only uniqueness of the solution to the Monge-Kantorovich problem, but also uniqueness of a gradient of a convex function $\nabla\varphi$ such that $\nabla\varphi\#\mu = \nu$.

5. We now prove that $\mathrm{Supp}(\nu) = \overline{\nabla\varphi(\mathrm{Supp}(\mu))}$. Let $x \in \mathrm{Supp}(\mu)$ be a differentiability point of φ, and let $y = \nabla\varphi(x)$. From (2.18) we know that for any $\varepsilon > 0$ there exists $\delta > 0$ with $\nabla\varphi(B_\delta(x)) \subset B_\varepsilon(y)$, and in particular

$$\nu[B_\varepsilon(y)] \geq \mu[\nabla\varphi^{-1}(\nabla\varphi(B_\delta(x)))] \geq \mu[B_\delta(x)].$$

But $\mu[B_\delta(x)] > 0$, for x lies in the support of μ; therefore $\nu[B_\varepsilon(y)] > 0$ too. Since ε is arbitrarily small, we deduce that $y \in \mathrm{Supp}(\nu)$. We conclude that

$$\nabla\varphi(\mathrm{Supp}(\mu)) \subset \mathrm{Supp}(\nu). \tag{2.43}$$

6. On the other hand, $\nu[\nabla\varphi(\mathrm{Supp}(\mu))] \geq \mu[\mathrm{Supp}(\mu)] = 1$. So ν is concentrated on $\nabla\varphi(\mathrm{Supp}\,\mu)$, and therefore (by definition of the support)

$$\mathrm{Supp}(\nu) \subset \overline{\nabla\varphi(\mathrm{Supp}(\mu))}.$$

Combining this with (2.43), we deduce that $\mathrm{Supp}(\nu)$ coincides with the closure of $\nabla\varphi(\mathrm{Supp}(\mu))$.

7. Statement (iii) is obvious by now.

8. Since π is optimal, we know that $d\pi(x,y)$-almost everywhere, $y = \nabla\varphi(x)$, which is equivalent to $x \in \partial\varphi^*(y)$. But since φ^* is finite $d\nu$-almost everywhere, it is also differentiable $d\nu$-almost everywhere. Hence, $d\pi(x,y)$-almost everywhere, $x = \nabla\varphi^*(y) = \nabla\varphi^*(\nabla\varphi(x))$. Taking the marginal, we find that this event holds true $d\mu(x)$-almost everywhere. The other assertion follows by symmetry. □

Exercise 2.14 (The Monge problem needs some regularity). This exercise was borrowed from [**15**]. Let $Q = [0,1]^{n-1}$. Let μ be the uniform probability measure on $Q \times \{0\}$, and ν the uniform probability measure on $Q \times \{-1\} \cup Q \times \{1\}$. Consider the cost function $c(x,y) = |x-y|^2$, or more generally $c(x,y) = c(|x-y|)$, where $c : \mathbb{R}_+ \to \mathbb{R}_+$ is a strictly increasing function.

(i) Show that the only solution to the Kantorovich problem is $\pi = (\delta_{x_1=1} + \delta_{x_1=-1})/2$, i.e. one should split each unit of mass in two halves and transport them along straight lines.

(ii) Deduce that there is no solution to the Monge problem.

Exercise 2.15 (Quantitative Knott-Smith criterion). The proof of part (i) of Theorem 2.12 is robust enough to accomodate small errors. Consider for instance the situation in which one is given a transference plan whose support is *almost* contained in the subdifferential of a convex function: does it follow that this transference plan is almost optimal? As an example, prove the following variant of Theorem 2.12 (i):

Theorem 2.16 (Knott-Smith optimality criterion, quantitative version). *Let μ and ν be probability measures on $\mathbb{R}^n$ with finite second order moments, let $\pi \in \Pi(\mu,\nu)$ and let φ be a proper lower semi-continuous convex*

function on $\mathbb{R}^n$ such that

$$\int_{\mathbb{R}^n\times\mathbb{R}^n} \big(\varphi(x)+\varphi^*(y)-x\cdot y\big)\,d\pi(x,y)\le\varepsilon.$$

Then, with the notation (2.1),

$$I[\pi]\le\left(\inf_{\Pi(\mu,\nu)} I\right)+\varepsilon.$$

Important Exercise 2.17 (Stability of the optimal transference mapping). Let $P_{\mathrm{ac},2}(\mathbb{R}^n)$ stand for the space of absolutely continuous probability measures with finite moments of order 2. Let $\sigma\in P_{\mathrm{ac},2}(\mathbb{R}^n)$ be given, and let ρ_k be a family of elements in $P_{\mathrm{ac},2}(\mathbb{R}^n)$, converging weakly to some $\rho\in P_{\mathrm{ac},2}(\mathbb{R}^n)$. Let $\nabla\varphi_k$ (resp. $\nabla\varphi$) be the optimal mapping in the Monge problem with quadratic cost, between σ and ρ_k (resp. ρ), and let π_k (resp. π) be the associated optimal transference plan. Our goal is to prove that *as $k\to\infty$, $\nabla\varphi_k$ converges in measure, relative to σ, to $\nabla\varphi$.* This means

$$\forall\varepsilon>0,\qquad \sigma[|\nabla\varphi_k-\nabla\varphi|\ge\varepsilon]\xrightarrow[k\to\infty]{}0.$$

As a general rule, convergence in measure implies convergence almost everywhere, however only up to extraction of a subsequence.

(i) Show that π_k converges weakly to π as $k\to\infty$.

(ii) Assuming that $\nabla\varphi$ is continuous, prove that $\nabla\varphi_k$ converges in measure to $\nabla\varphi$.

(iii) Prove the desired conclusion without assuming the continuity of $\nabla\varphi$, thanks to **Lusin's theorem**: *if f is a bounded measurable function on a compact set K, then for any $\varepsilon>0$ there exists a continuous function f_ε on K such that $\|f_\varepsilon\|_{L^\infty}=\|f\|_{L^\infty}$, and f coincides with f_ε outside of a set of measure at most ε.*

Hints: (i) Use the fact that the optimal transference plan is unique. (ii) If $\nabla\varphi$ is continuous, then the set $A_\varepsilon=\{(x,y);\,|y-\nabla\varphi(x)|<\varepsilon\}$ is open (why?), so $1=\pi[A_\varepsilon]\le\liminf\pi_k[A_\varepsilon]$. (iii) First find M_ε such that $\sigma[|\nabla\varphi|\ge M_\varepsilon]<\varepsilon/3$; then R_ε such that $\sigma[|x|>R_\varepsilon]<\varepsilon/3$; then apply Lusin's theorem on the compact set $B(0,R_\varepsilon)$ to some truncation of $\nabla\varphi$. A detailed argument can be found, in a slightly different context, in [**15**]. A proof of Lusin's theorem can be found in [**231**].

2.1.6. Duality worked out by hand. We now make an important remark: one can dispense with an explicit use of the Kantorovich duality, and rather *recover it* by working out the Euler-Lagrange equation associated with the dual optimization problem. This strategy has been carefully implemented by Gangbo [**148**]. Also Caffarelli [**74**] explains this approach, and gives credit to Varadhan for it. It is often useful to keep this remark in

mind when one wishes to address other variational problems for which there is not such a good duality theory as for the Monge-Kantorovich problem.

The method: Let μ, ν be probability measures on $\mathbb{R}^n$, with respective supports X and Y. Let (φ, φ^*) be an optimizing pair of the dual minimization problem, with φ convex lower semi-continuous on $\mathbb{R}^n$. Recall that we have constructed this pair in Theorem 2.9, without any mention of the Kantorovich duality. We are going to perform some variations of this optimizer. Whenever t is a (small) positive number, and h a bounded continuous function on $\mathbb{R}^n$, one can introduce the pair of dual functions $((\varphi^* + th)^*, \varphi^* + th) \in \widetilde{\Phi}$. (Note: $\varphi^* + th$ needs not be convex, but this does not matter.) Since (φ, φ^*) is optimal,

$$J\big((\varphi^* + th)^*, \varphi^* + th\big) \geq J(\varphi, \varphi^*),$$

which can be rewritten as

$$0 \leq \int_Y h\, d\nu + \int_X \left[\frac{(\varphi^* + th)^* - \varphi}{t}\right] d\mu. \tag{2.44}$$

We wish to pass to the limit in this inequality as $t \downarrow 0$. Assume for simplicity that X, Y are compact sets, and denote by y_t a point where the function

$$y \longmapsto x \cdot y - (\varphi^* + th)(y)$$

achieves its maximum. Of course y_t is a function of x, and $y_0 = \nabla\varphi(x)$. One can check that y_t converges to y_0 as $t \downarrow 0$. Then, the reader can prove as an exercise that, for all $x \in \mathrm{Int}(X)$, and $t > 0$ small enough,

$$-h(y_0) \leq \frac{(\varphi^* + th)^*(x) - \varphi(x)}{t} \leq -h(y_t).$$

(**Hint:** Express $(\varphi^* + th)^*(x)$ in terms of y_t on one hand, $\varphi = \varphi^{**}$ in terms of y_0 on the other hand).

By Lebesgue's dominated convergence theorem, one can then pass to the limit in (2.44), and recover

$$0 \leq \int_Y h\, d\nu - \int_X h(\nabla\varphi(x))\, d\mu(x).$$

Replacing h by $-h$, we conclude that actually

$$\int_X (h \circ \nabla\varphi)\, d\mu = \int_Y h\, d\nu.$$

Since this holds true for arbitrary bounded continuous h, it follows that $\nabla\varphi\#\mu = \nu$. Once this is obtained, it is not difficult to conclude by the same lines as above. □

2.2. The real line

Before discussing alternative ways towards the optimal transportation theorem, let us study the very peculiar, but important case of the real line.

On $\mathbb{R}$, gradients of convex functions coincide with nondecreasing functions. And subgradients are "complete nondecreasing graphs", or maximal monotone subsets of $\mathbb{R}^2$. By definition, a subset Γ of $\mathbb{R}^2$ is said to be monotone if

$$(x_1, y_1), (x_2, y_2) \in \Gamma \Longrightarrow \quad [x_1 \leq x_2 \text{ and } y_1 \leq y_2] \text{ or } [x_1 \geq x_2 \text{ and } y_1 \geq y_2] \tag{2.45}$$

(note that this is the same as $(x_1 - x_2)(y_1 - y_2) \geq 0$). Geometrically, a complete nondecreasing graph is nothing but the usual graph of a nondecreasing function, with possibly some vertical lines added to make this graph "continuous" (see Figure 2.4). These lines correspond to the points x where the left and right derivatives $\varphi'_-(x)$ and $\varphi'_+(x)$ of the convex function φ do not agree. Indeed, for any $x \in \mathbb{R}$,

$$\partial\varphi(x) = [\varphi'_-(x), \varphi'_+(x)].$$

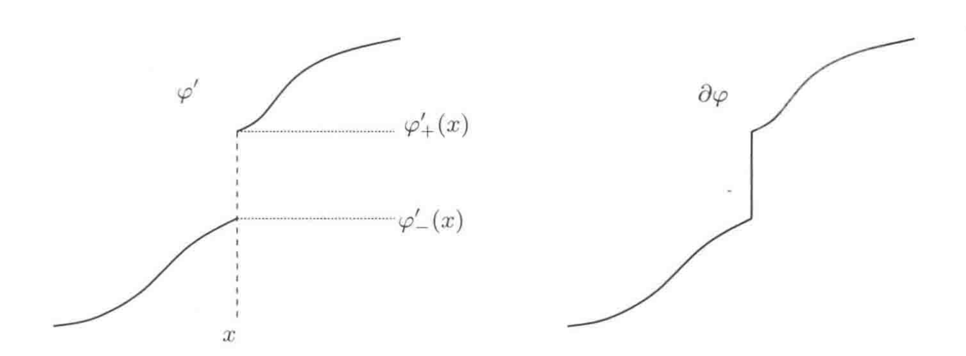


Figure 2.4. Derivative vs. subdifferential

2.2.1. Cumulative distribution functions.

Any probability measure μ on $\mathbb{R}$ can be represented by its distribution function, or, in a less ambiguous terminology, its **cumulative distribution function**:

$$F(x) = \int_{-\infty}^{x} d\mu = \mu\big[(-\infty, x]\big].$$

Recall from elementary probability theory that F is *right*-continuous, nondecreasing, and $F(-\infty) = 0$, $F(+\infty) = 1$.

Then, one can define the **generalized inverse** of F on $[0, 1]$ by

$$F^{-1}(t) = \inf\big\{x \in \mathbb{R};\ F(x) > t\big\}.$$

The generalized inverse of F is also right-continuous (see Figure 2.5). Note the inequalities

$$\forall x \in \mathbb{R}, \quad F^{-1}(F(x)) \geq x; \qquad \forall t \in [0,1], \quad F(F^{-1}(t)) \geq t.$$

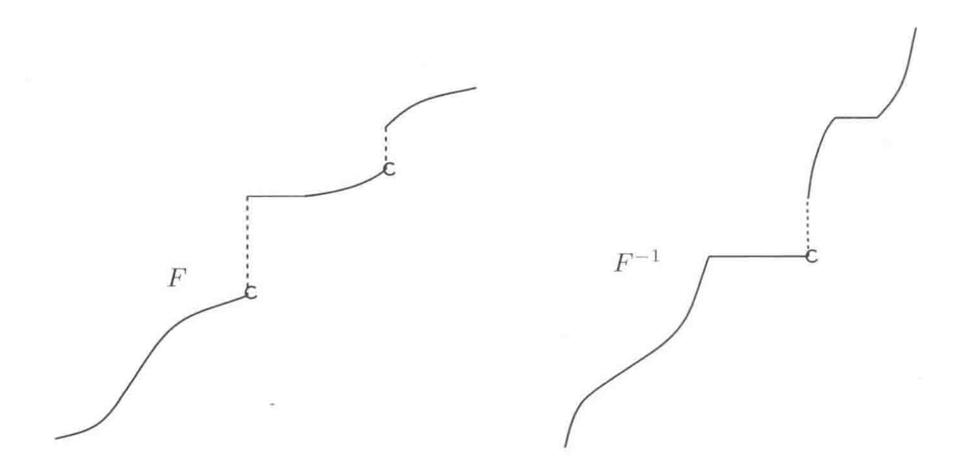


Figure 2.5. Generalized inverse

As for probability measures on the product space $\mathbb{R} \times \mathbb{R}$, they can be represented by joint two-dimensional cumulative distribution functions:

$$H(x_0, y_0) = \int_{R(x_0,y_0)} d\pi = \pi[R(x_0, y_0)],$$

where $R(x_0, y_0)$ is the rectangle made of all points $(x, y) \in \mathbb{R}^2$ with $x \leq x_0$, $y \leq y_0$. We will use the notation $d\pi = dH$. A function H on $\mathbb{R}^2$ which is nondecreasing, right-continuous in both variables x and y, and has limits 0 and 1 at $(-\infty, -\infty)$ and $(+\infty, +\infty)$ respectively, gives rise to a unique probability measure on $\mathbb{R}^2$. To see this, note that H determines the mass of every rectangle with sides parallel to the coordinate axes, and recall that such rectangles generate all Borel sets in $\mathbb{R}^2$.

2.2.2. Optimal transportation theorem on the line. We can now state the solution to the optimal transportation problem in terms of cumulative distribution functions. The following theorem was already known to Hoeffding and Fréchet.

Theorem 2.18 (Optimal transportation theorem for a quadratic cost on $\mathbb{R}$). *Let μ, ν be two probability measures on $\mathbb{R}$, with respective cumulative distribution functions F and G. Let π be the probability measure*

on $\mathbb{R}^2$ with joint two-dimensional cumulative distribution function

$$H(x,y) = \min(F(x), G(y)). \tag{2.46}$$

Then, π belongs to $\Pi(\mu,\nu)$, and is optimal in the Kantorovich transportation problem between μ and ν for the quadratic cost function $c(x,y) = |x-y|^2$. Moreover, the value of the optimal transportation cost is

$$\mathcal{T}_2(\mu,\nu) = \int_0^1 |F^{-1}(t) - G^{-1}(t)|^2 \, dt. \tag{2.47}$$

Remarks 2.19. (i) The **Hoeffding-Fréchet theorem** states that a nonnegative function H on $\mathbb{R}^2$, nondecreasing and right-continuous in each argument, defines a probability measure π on $\mathbb{R}^2$ with given marginals μ,ν if and only if

$$\forall (x,y) \in \mathbb{R}^2, \qquad F(x) + G(y) - 1 \leq H(x,y) \leq \min[F(x), G(y)],$$

where F and G are the cumulative distribution functions associated with μ and ν respectively. For background on this theorem, see for instance [**225**].

(ii) In fact, the π constructed in Theorem 2.18 is optimal *whatever the convex cost.* More precisely, π is optimal as soon as the cost function $c(x,y)$ takes the form $c(x-y)$, where c is a convex nonnegative symmetric function on $\mathbb{R}$. In this case the optimal transportation cost is

$$\mathcal{T}_c(\mu,\nu) = \int_0^1 c\big(F^{-1}(t) - G^{-1}(t)\big) \, dt.$$

In the case of a concave cost function, the situation is completely different: see the explanations and references at the end of this chapter.

(iii) In particular, the total transportation cost associated with the cost function $c(x,y) = |x-y|$ is

$$\mathcal{T}_1(\mu,\nu) = \int_0^1 |F^{-1}(t) - G^{-1}(t)| \, dt = \int_{\mathbb{R}} |F(x) - G(x)| \, dx, \tag{2.48}$$

where the second inequality is a consequence of Fubini's theorem (exercise!). Thus, in this case, the optimal transportation cost coincides with the L^1 distance between cumulative distribution functions.

(iv) If μ does not give mass to points, then $T = G^{-1} \circ F$ transports μ onto ν, and

$$\int_{-\infty}^{x} d\mu = \int_{-\infty}^{T(x)} d\nu. \tag{2.49}$$

This expresses the general fact that the solution to the transportation problem is given by the **monotone rearrangement** of μ onto ν (one proceeds to transfer the sand into the hole *starting from the left.*) Note that discontinuity points of G correspond to atoms for ν. Whenever ν has an atom,

$G^{-1} \circ F$ will then be constant on some interval (when encountering an atom in the filling process, one must keep putting mass in this atom for some time).

(v) Assume that μ, ν have respective densities f and g with respect to Lebesgue measure, that f and g are continuous, and that g is strictly positive. Then, T is a C^1 function, and by differentiating (2.49), we get the basic, important identity

$$f(x) = g(T(x))\, T'(x). \tag{2.50}$$

With the preceding remarks and formulas, it is in principle possible to compute all the relevant quantities in the transportation problem on the real line, for a quadratic cost function, or a more general convex cost function. This should be kept in mind when one tries to check, or disprove, general statements about optimal transportation. Also, this makes it possible to compute optimal transportation in many cases where the distribution and the cost function are all radially symmetric.

Proof of Theorem 2.18. There are several elementary ways of proving this theorem (see [**225**]). However, we prefer to derive it as a corollary of the general optimal transportation theorem in arbitrary dimension. Whenever F is a cumulative distribution function, we shall denote by $F(x^-)$ the left-limit $\lim_{z \uparrow x} F(z)$; by monotonicity, this limit always exists. Since F is right-continuous, there is no need to introduce right-limits.

1. We claim that

$$\mathrm{Supp}(\pi) \subset \Big\{(x,y) \in \mathbb{R}^2; \quad F(x^-) \leq G(y) \text{ and } G(y^-) \leq F(x)\Big\}. \tag{2.51}$$

To prove this, assume for instance $F(x^-) > G(y)$. From the right-continuity of G and the fact that both F and G are nondecreasing functions, we deduce that if x' belongs to a small neighborhood of x and y' to a small neighborhood of y, then $F(x') > G(y')$. So,

$$H(x', y') = \min\,[F(x'), G(y')] = G(y').$$

Thus, on a small rectangle centered at (x, y), the function H does not depend on the first variable x'. This easily entails that $d\pi = dH$ assigns zero mass to this rectangle, so $(x, y) \notin \mathrm{Supp}(\pi)$.

Then, one has to be careful! It does *not* follow that $G^{-1} \circ F(x^-) \leq y \leq G^{-1} \circ F(x)$, as Figure 2.6 demonstrates.

But one can check that the monotonicity condition (2.45) holds on the support of π. Indeed, let (x_1, y_1), (x_2, y_2) be two points in the support of π. Assume for instance $x_1 > x_2$; we have to check that $y_1 \geq y_2$. Applying (2.51)

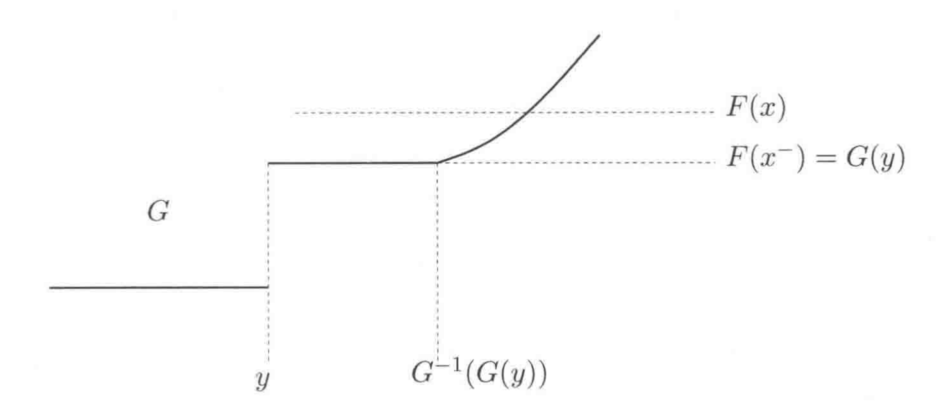


Figure 2.6. A situation in which $y < G^{-1} \circ F(x^-) = G^{-1}(G(y))$.

and using the fact that F is nondecreasing, we find that

$$G(y_1) \geq F(x_1^-) \geq F(x_2) \geq G(y_2^-).$$

If $G(y_1) > G(y_2^-)$, then $y_2 \leq y_1$, and we are done.

Otherwise, necessarily $G(y_1) = F(x_1^-) = F(x_2) = G(y_2^-)$. If $y_2 > y_1$, this means that F is constant on $[x_2, x_1)$, and G on $[y_1, y_2)$. Let us prove that this case is impossible, in the sense that (x_1, y_1) and (x_2, y_2) *cannot* belong to the support of π. We consider for instance (x_2, y_2). Introduce a small number $\varepsilon > 0$; we shall prove that the rectangle R whose endpoints have respective first and second coordinates equal to $(x_2 - \varepsilon, y_2 - \varepsilon)$ and $(x_2 + \varepsilon, y_2 + \varepsilon)$ has zero measure for π. For this we express its measure in terms of H:

$$\pi[R] = H(x_2+\varepsilon, y_2+\varepsilon)+H(x_2-\varepsilon, y_2-\varepsilon)-H(x_2-\varepsilon, y_2+\varepsilon)-H(x_2+\varepsilon, y_2-\varepsilon).$$

Using the definition of H, the inequalities $x_2 < x_1$ and $y_2 > y_1$, and the nondecreasing property of F and G, it is a simple exercise to show that all terms in the preceding expression cancel if ε is chosen small enough; then $\pi[R] = 0$. This shows that the assumption $y_2 > y_1$ is impossible, and so (2.51) is true.

2. Thus π has its support included in a monotone subset of $\mathbb{R}^2$, hence in the subdifferential of a lower semi-continuous convex function. By the Knott-Smith optimality criterion (Theorem 2.12 (i)), π is an optimal transference plan. Next, we claim that

$$\pi = (F^{-1} \times G^{-1})\# \mathcal{L}, \qquad (2.52)$$

where $\mathcal{L}$ stands for the Lebesgue measure in $[0, 1]$. Indeed, it suffices to check this identity on an arbitrary rectangle of the form $R(x, y)$, and then (2.52)

becomes

$$\pi[R(x,y)] = \Big|\{(F^{-1}(t), G^{-1}(t)) \in R(x,y)\}\Big|$$

(with the notation $|A| = \mathcal{L}[A]$). The last quantity is

$$\Big|\{t \in \mathbb{R};\ F^{-1}(t) \le x\} \cap \{t \in \mathbb{R};\ G^{-1}(t) \le y\}\Big|. \tag{2.53}$$

Depending on the cases, $\{F^{-1}(t) \le x\}$ is $[0, F(x)]$ or $[0, F(x))$; anyway the set whose Lebesgue measure is taken in (2.53) is an interval with endpoints 0 and $\min[F(x), G(y)]$, and its measure is equal to $\min[F(x), G(y)] = H(x,y)$. This proves the claim.

3. As a consequence of (2.52), for any nonnegative measurable function u on $\mathbb{R}^2$,

$$\int_{\mathbb{R}^2} u(x,y)\, d\pi(x,y) = \int_0^1 u(F^{-1}(t), G^{-1}(t))\, dt.$$

This implies (2.47), and concludes the proof of Theorem 2.18. □

Exercise 2.20 (Degeneracy of the distance cost function). Let μ, ν be probability measures on $\mathbb{R}$, such that there exist disjoint intervals I, J in $\mathbb{R}$, with $\operatorname{Supp} \mu \subset I$, $\operatorname{Supp} \nu \subset J$. Consider the cost function $c(x,y) = |x-y|$ and show that any $\pi \in \Pi(\mu, \nu)$ is optimal.

Hint: One possibility is to note that $\varphi(x) = \pm x$ achieves equality in the Kantorovich duality.

2.3. Alternative arguments

Now, we shall come back to the multi-dimensional case, and sketch some ideas of a geometrical nature, by which one can recover the results of Section 2.1 — sometimes with more generality. They are mainly based on the concept of **cyclical monotonicity**. This approach was already suggested by Knott and Smith, as well as Brenier. Later, it was generalized to an abstract setting by Rüschendorf [**233**] and Rachev and Rüschendorf [**225**], while McCann [**202**] and Gangbo and McCann [**151**] took full advantage of this method in $\mathbb{R}^n$.

In this section we shall not give complete proofs, but rather try to convey the ideas involved. Again, we focus on the case of a quadratic cost function, deferring generalizations to Section 2.4. As compared to the duality method, the techniques discussed below have at least two advantages:

- they rely on appealing geometrical ideas;

- they do not require any moment condition on μ, ν.

In fact, the Kantorovich duality can be recovered as a *consequence* of these methods.

The proofs will rely on two main tools. The first is Rockafellar's theorem [**227**] about the characterization of so-called cyclically monotone sets. The second is a lemma introduced by Aleksandrov [**7**] to prove uniqueness for the solution of the equation of prescribed Gauss curvature (see Chapter 4), and generalized by McCann. These two statements will be presented in subsections 2.3.2 and 2.3.3, respectively. Before that, we shall introduce the basic notions about cyclical monotonicity.

2.3.1. Cyclical monotonicity. We begin with a (hopefully) enlightening exercise.

Exercise 2.21 (The discrete case). Let $x_1, \ldots, x_N$ and $y_1, \ldots, y_N$ be points in $\mathbb{R}^n$ (not necessarily distinct). Let $\mu = (1/N)\sum_{i=1}^N \delta_{x_i}$, $\nu = (1/N)\sum_{j=1}^N \delta_{y_j}$, and $\pi = (1/N)\sum_{i=1}^N \delta_{(x_i,y_i)}$. We denote by $\mathcal{S}_N$ the set of permutations of $\{1, \ldots, N\}$.

(i) Show that π is optimal in the Monge-Kantorovich problem of transporting μ onto ν with quadratic cost if and only if for all permutations $\sigma \in \mathcal{S}_N$,

$$\sum_{i=1}^N |x_i - y_i|^2 \le \sum_{i=1}^N |x_i - y_{\sigma(i)}|^2. \tag{2.54}$$

(ii) Show that (2.54) is equivalent to the following requirement: for all $m \le N$ and for all $i_1, \ldots, i_m$ in $\{1, \ldots, N\}$,

$$\sum_{k=1}^m |x_{i_k} - y_{i_k}|^2 \le \sum_{k=1}^m |x_{i_k} - y_{i_{k-1}}|^2, \tag{2.55}$$

where $i_0 = i_m$ by convention.

(iii) Show that (2.55) is equivalent to

$$\sum_{k=1}^m y_{i_k} \cdot (x_{i_{k+1}} - x_{i_k}) \le 0, \tag{2.56}$$

with the convention $i_{m+1} = i_1$.

Hints: (i) You may use the Birkhoff and Krein-Milman theorems (recalled in Section 1 of the Introduction) to write any transference plan as an average of "permutations". (ii) Recall that any permutation can be decomposed into cycles with disjoint supports.

It turns out that the condition (2.56) is well-known in convex analysis.

Definition 2.22 (Cyclical monotonicity). A subset $\Gamma \subset \mathbb{R}^n \times \mathbb{R}^n$ is said to be cyclically monotone if it fulfills the following condition: for all $m \ge 1$,

and for all $(x_1, y_1), \ldots, (x_m, y_m)$ in Γ,

$$\sum_{i=1}^{m} |x_i - y_i|^2 \leq \sum_{i=1}^{m} |x_i - y_{i-1}|^2, \tag{2.57}$$

with the convention $y_0 = y_m$, or equivalently

$$\sum_{i=1}^{m} y_i \cdot (x_{i+1} - x_i) \leq 0, \tag{2.58}$$

with the convention $x_{m+1} = x_1$.

Remark 2.23. If $\pi_k \to \pi$ weakly, then every point (x, y) in the support of π can be approached by a sequence (x_k, y_k) in the support of π_k. This implies that a weak limit of probability measures with cyclically monotone support also has cyclically monotone support. In other words, *cyclical monotonicity is stable under weak convergence.*

Proposition 2.24 (Optimal plans have cyclically monotone support). *Let μ, ν be two probability measures on $\mathbb{R}^n$, and let $\pi \in \Pi(\mu, \nu)$ be optimal in the Kantorovich problem of mass transference between μ and ν, with quadratic cost $c(x, y) = |x - y|^2$. Then, the support of π is cyclically monotone.*

Before we sketch the proof of this proposition, we make an important comment: it is not really a "characterization" of optimal transference plans. In order to get such a characterization, one should prove the following converse property:

Open Problem 2.25 (Optimality criterion for quadratic cost). *Let μ, ν be two probability measures on $\mathbb{R}^n$, and let $\pi \in \Pi(\mu, \nu)$ with cyclically monotone support. Then, π is optimal in the Kantorovich problem of mass transference between μ and ν, with quadratic cost $c(x, y) = |x - y|^2$.*

Even though this statement sounds very likely, it has apparently never been proven. When μ does not give mass to small sets, it is a direct consequence of the uniqueness statement in Theorem 2.32. Otherwise, the closest known result is a theorem by Ambrosio (see [**15**]), stating that cyclical monotonicity of the support is indeed a sufficient condition for optimality, provided that

$$\int_{\mathbb{R}^n \times \mathbb{R}^n} |x - y|^2 \, d\mu(x) \, d\nu(y) < +\infty.$$

Let us give an informal sketch of a proof of Proposition 2.24; an alternative proof is suggested in Exercise 2.38.

Sketch of proof of Proposition 2.24. Let π be optimal; we wish to show that its support is cyclically monotone. Let $(x_1, y_1), \ldots, (x_m, y_m)$ be m points in the support of π (in particular, $x_1, \ldots, x_m$ have to belong to the support of μ, and $y_1, \ldots, y_m$ to the support of ν). Assume that $\sum |x_i - y_i|^2 > \sum |x_i - y_{i-1}|^2$, and cut small balls B_i, centered at (x_i, y_i), each of them carrying a positive mass ε of the measure π. Then redefine π by shifting each ball B_i to a new position centered at the point (x_i, y_{i-1}) (see Figure 2.7). Let $\widetilde{\pi}$ be the new measure thus defined. The X-marginal of $\widetilde{\pi}$ is still μ; on the other hand, the Y-marginal of $\widetilde{\pi}$ is still approximatively ν, because at each point y_i the mass which has been removed by the displacement of B_i has been compensated for by the displacement of B_{i+1} (with the convention $B_{m+1} = B_1$). On the other hand, the total cost associated with $\widetilde{\pi}$ is strictly less than the total cost associated with $\widetilde{\pi}$, the difference being approximately

$$\varepsilon \left(\sum_{i=1}^{N} |x_i - y_i|^2 - \sum_{i=1}^{N} |x_i - y_{i-1}|^2 \right).$$

This shows that π is not optimal.

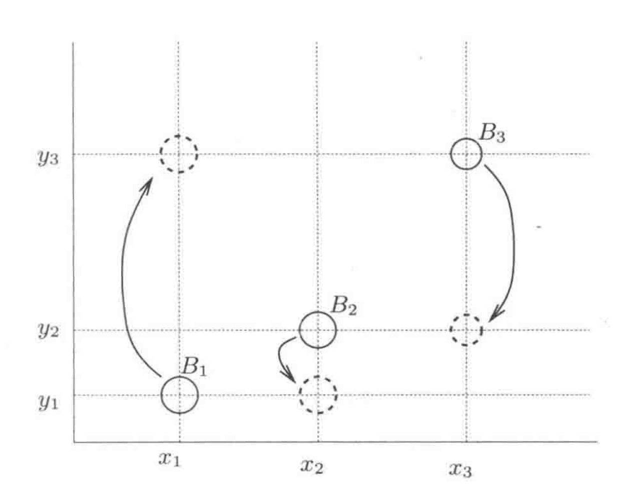


Figure 2.7. Cyclical permutation of small balls

Of course, this proof is not rigorous, because the second marginal of $\widetilde{\pi}$ is not exactly ν. However, it is not very difficult to turn it into a rigorous argument, see [**151**]. □

Remarks 2.26. (i) There is nothing special in these arguments with the quadratic cost. The same reasoning would work for any lower semi-continuous cost function c, provided that $|x - y|^2$ in Definition 2.57 is replaced by $c(x, y)$. See also subsection 2.4.1.

(ii) If one combines the density of convex combinations of Dirac masses in $P(\mathbb{R}^n)$ with the stability of cyclical monotonicity under weak convergence, one immediately obtains the existence of at least one optimal transference plan whose support is cyclically monotone. This remark is due to McCann [**202**].

In the next subsection, we shall link Proposition 2.24 with the already encountered characterization of optimal transference plans by the Knott-Smith criterion.

2.3.2. Rockafellar's theorem. The main theorem about cyclical monotonicity is

Theorem 2.27 (Rockafellar's theorem). *A nonempty subset* $\Gamma \subset \mathbb{R}^n \times \mathbb{R}^n$ *is cyclically monotone if and only if it is included in the subdifferential of a proper lower semi-continuous convex function* φ *on* $\mathbb{R}^n$*. Moreover, maximal (with respect to inclusion) cyclically monotone subsets are exactly the subdifferentials of proper lower semi-continuous convex functions.*

Of course, in this statement we have identified the subdifferential with its graph.

Proof of Theorem 2.27. The proof is surprisingly elementary.

1. Let φ be a convex function; we shall prove that its subdifferential is a cyclically monotone subset of $\mathbb{R}^n \times \mathbb{R}^n$. Of course, it will follow that every subset of the subdifferential is cyclically monotone.

So let $(x_1, y_1), \ldots, (x_m, y_m)$ be such that for all i $(1 \leq i \leq m)$, $y_i \in \partial\varphi(x_i)$. By definition, this means that for all $z \in \mathbb{R}^n$,

$$\varphi(z) \geq \varphi(x_i) + y_i \cdot (z - x_i).$$

In particular,

$$\begin{cases} \varphi(x_2) \geq \varphi(x_1) + y_1 \cdot (x_2 - x_1), \\ \varphi(x_3) \geq \varphi(x_2) + y_2 \cdot (x_3 - x_2), \\ \quad \vdots \\ \varphi(x_1) \geq \varphi(x_m) + y_m \cdot (x_1 - x_m). \end{cases}$$

Adding up all these inequalities, we recover (2.58).

2. Now, let Γ be a cyclically monotone subset of $\mathbb{R}^n \times \mathbb{R}^n$; we shall construct a proper lower semi-continuous convex function φ such that $\Gamma \subset \text{Graph}\,(\partial\varphi)$, and this will end the proof of the theorem.

Pick any $(x_0, y_0) \in \Gamma$, and define

$$\varphi(x) = \sup \Big\{ y_m \cdot (x - x_m) + \cdots + y_0 \cdot (x_1 - x_0); \quad m \in \mathbb{N}, \\ (x_1, y_1), \ldots, (x_m, y_m) \in \Gamma \Big\}. \tag{2.59}$$

Since φ is a supremum of affine functions, it is a lower semi-continuous convex function. Moreover, by cyclical monotonicity, $\varphi(x_0) \leq 0$ (actually $\varphi(x_0) = 0$: in the definition, choose $m = 1$, $x_1 = x_0$, $y_1 = y_0$). So φ is proper. *This is the only place where cyclical monotonicity is used!*

Now let us prove that Γ is included in the graph of the subdifferential, $\mathrm{Graph}\,(\partial\varphi)$. Let $(x, y) \in \Gamma$; we have to check that

$$\forall z \in \mathbb{R}^n, \qquad \varphi(z) \geq \varphi(x) + y \cdot (z - x).$$

So pick an arbitrary $z \in \mathbb{R}^n$. Obviously it suffices to check that for all $\alpha < \varphi(x)$,

$$\varphi(z) \geq \alpha + y \cdot (z - x). \tag{2.60}$$

If $\alpha < \varphi(x)$, then by the definition of φ there exist m and $x_1, y_1, \ldots, x_m, y_m$ such that

$$\alpha \leq y_m \cdot (x - x_m) + \cdots + y_0 \cdot (x_1 - x_0).$$

Thus,

$$\alpha + y \cdot (z - x) \leq y \cdot (z - x) + y_m \cdot (x - x_m) + \cdots + y_0 \cdot (x_1 - x_0).$$

Set $x = x_{m+1}$, $y = y_{m+1}$ and apply the definition of φ, to obtain (2.60).

3. It remains to handle the maximality. If Γ is a maximal cyclically monotone subset of $\mathbb{R}^n \times \mathbb{R}^n$, then it is included in a subdifferential $\partial\varphi$, so by maximality $\Gamma = \partial\varphi$. Conversely, if $\partial\varphi$ is included in a cyclically monotone set Γ, then $\Gamma \subset \partial\widetilde{\varphi}$ for some function φ, so $\partial\varphi \subset \partial\widetilde{\varphi}$. This implies $\nabla\varphi = \nabla\widetilde{\varphi}$, and by taking limits and closed convex hulls, $\partial\varphi = \partial\widetilde{\varphi}$. □

Remark 2.28. A lower semi-continuous convex function is determined by its subdifferential only up to an additive constant.

From Proposition 2.24 and Theorem 2.27 we at once deduce

Theorem 2.29 (Optimal plans are supported in subdifferentials). *Let μ, ν be probability measures on $\mathbb{R}^n$, and let $\pi \in \Pi(\mu, \nu)$ be a transference plan. If π is optimal for the Kantorovich problem with quadratic cost, $c(x, y) = |x - y|^2$, then π is supported in the subdifferential of a proper lower semi-continuous convex function.*

In this theorem, we did not take advantage of the Euclidean setting, and the same theorem would be true on an arbitrary Hilbert space, for instance. If now one takes advantage of the differentiability properties of

convex functions on $\mathbb{R}^n$, one can derive a more precise result. This was used by McCann to obtain

Corollary 2.30 (Brenier's theorem, refined version; existence part). *Let μ and ν be probability measures on $\mathbb{R}^n$, such that μ does not give mass to small sets. Then there exists a convex function φ on $\mathbb{R}^n$ such that*

$$\nabla\varphi\#\mu = \nu.$$

Note that the assumptions here are more general than those in Theorem 2.12, since the finiteness of the second order moments is not required.

To complete the picture, it remains to understand how to prove uniqueness of the optimal transference plan, under the usual regularity assumption on μ, without going through the duality theory. This is the object of the next subsection.

2.3.3. Aleksandrov's lemma.

In [**202**], McCann noticed that the uniqueness theorem could be deduced from the following lemma, due to Aleksandrov [**7**] and revisited by McCann himself.

Lemma 2.31 (Aleksandrov's lemma). *Let φ and $\overline{\varphi}$ be two convex functions such that $\varphi(x_0) = \overline{\varphi}(x_0)$, but $\nabla\varphi(x_0) \neq \nabla\overline{\varphi}(x_0)$. Let $V = \{\varphi > \overline{\varphi}\}$, and*

$$Z = \nabla\overline{\varphi}^{-1}\big(\nabla\varphi(V)\big).$$

Then $x_0 \in \overline{V}$, $Z \subset V$, but Z lies a positive distance away from x_0.

We refer to [**202**] for the proof, and to Figure 2.8 for an illustration.

With the help of Lemma 2.31, McCann eventually managed to extend the validity of Brenier's theorem: he showed that the Brenier map is well-defined without any assumption of finite second order moment.

Theorem 2.32 (Brenier's theorem, refined version). *Let μ, ν be two probability measures on $\mathbb{R}^n$, such that μ does not give mass to small sets. Then there is exactly one measurable map T such that $T\#\mu = \nu$ and $T = \nabla\varphi$ for some convex function φ, in the sense that any two such maps coincide $d\mu$-almost everywhere.*

Proof of Theorem 2.32. We already proved existence earlier, so the problem is to prove uniqueness. Assume that there are two different convex functions φ, $\overline{\varphi}$ such that $\nabla\varphi\#\mu = \nabla\overline{\varphi}\#\mu = \nu$, and $\nabla\varphi$, $\nabla\overline{\varphi}$ are not identically equal on the support of μ. Let $x_0 \in \mathrm{Supp}(\mu)$ be such that $\nabla\varphi(x_0) \neq \nabla\overline{\varphi}(x_0)$; without loss of generality, one can assume that $\varphi(x_0) = \overline{\varphi}(x_0)$, and then by an adequate nonsmooth **implicit function theorem for convex functions**, proven in McCann [**202**, Appendix], one can show that the set defined by $\{\varphi = \overline{\varphi}\}$ is small. Since x_0 belongs to the support of μ and since μ

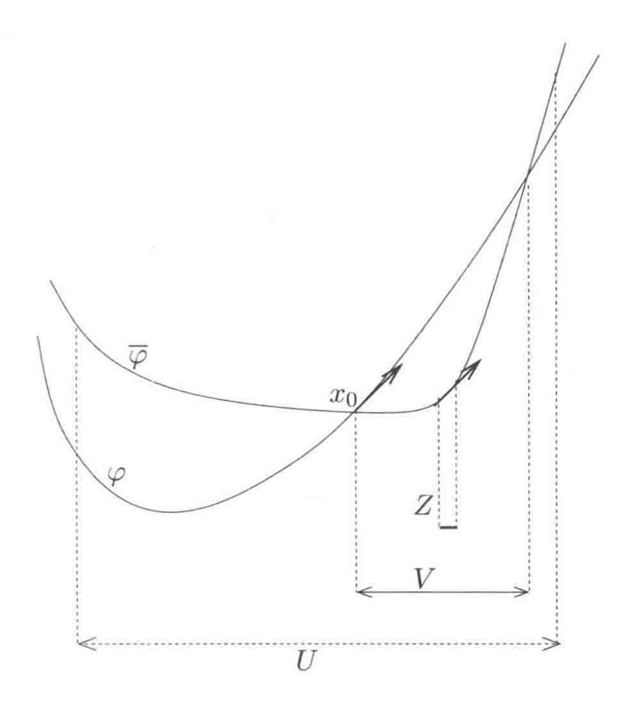


Figure 2.8. Schematic picture of Aleksandrov's lemma. Here U is a neighborhood of x_0 on which $(\varphi = \overline{\varphi})$ is the equation of a small set; $V = (\varphi > \overline{\varphi}) \cap U$; and $Z = \nabla\overline{\varphi}^{-1}(\nabla\varphi(V))$.

does not charge small sets, any sufficiently small (in the usual sense!) neighborhood of x_0 intersects either $\{\varphi > \overline{\varphi}\}$, or $\{\varphi < \overline{\varphi}\}$, in a set of positive measure for μ. Up to exchanging φ and $\overline{\varphi}$, assume that it is the former; then we can apply Lemma 2.31. Since x_0 lies a positive distance away from Z, it follows that Z avoids a set of positive measure for μ, so that

$$\mu[Z] < \mu[V].$$

This implies that $\nabla\varphi\#\mu \neq \nabla\overline{\varphi}\#\mu$. Indeed,

$$\begin{aligned}\nabla\overline{\varphi}\#\mu[\nabla\varphi(V)] = \mu[\nabla\overline{\varphi}^{-1}(\nabla\varphi(V))] = \mu[Z] < \mu[V] \leq \mu[\nabla\varphi^{-1}(\nabla\varphi(V))] \\ = \nabla\varphi\#\mu[\nabla\varphi(V)].\end{aligned}$$

The proof is complete. □

2.4. Generalizations to other costs

Generalizations of the optimal transportation theorem, based either on duality [**150**], or on geometrical arguments, work beautifully. The main idea, already found in Knott and Smith and in Brenier, is to use generalized notions of convexity, conjugation, subdifferentials, cyclical monotonicity,

etc. For complete developments of these notions, we refer to Rachev and Rüschendorf [**225**, Chapter 3.3] and the references therein, and to Gangbo and McCann [**151**]. The framework in [**225**] is extremely wide (measurable spaces with a very weak regularity condition), while [**151**] gives a more thorough study, including some regularity properties, in $\mathbb{R}^n$. Caffarelli [**77**] may also be consulted.

The case of a Riemannian manifold was studied in [**206**]; it has many common points with that of $\mathbb{R}^n$. In the sequel, we give a short account of the main results obtained by the above-mentioned authors (the order of presentation of the results has little to do with their chronological order of appearance).

2.4.1. General notions. To emphasize the generality of the notions which we are going to introduce, we consider a rather abstract framework.

Definition 2.33 (Generalized convexity notions). Let X and Y be two nonempty sets, and let $c(x, y)$ be defined on $X \times Y$, with values in $\mathbb{R} \cup \{+\infty\}$.

A function $\varphi : X \to \mathbb{R} \cup \{-\infty\}$ is said to be *c-concave* if there exists $\psi : Y \to \mathbb{R} \cup \{-\infty\}$, $\psi \not\equiv -\infty$, such that

$$\forall x \in X, \qquad \varphi(x) = \inf_{y \in Y} [c(x, y) - \psi(y)].$$

The *domain* of φ, $\mathrm{Dom}(\varphi)$, is defined as the set of points $x \in X$ such that $\varphi(x) \neq -\infty$.

A subset Γ of $X \times Y$ is said to be *c-cyclically monotone* if for any $m \geq 1$, and any family $(x_1, y_1), \ldots, (x_m, y_m)$ in Γ,

$$\sum_{i=1}^{m} c(x_i, y_i) \leq \sum_{i=1}^{m} c(x_i, y_{i-1}),$$

with the convention $y_0 = y_m$.

The *c-superdifferential* $\partial^c \varphi$ of a c-concave function φ is defined as the set of all pairs $(x, y) \in X \times Y$ such that

$$\forall z \in X, \qquad \varphi(z) \leq \varphi(x) + [c(z, y) - c(x, y)].$$

Finally, if φ is any function on X, with values in $\mathbb{R} \cup \{-\infty\}$ and not identically $-\infty$, one can define its *c-transform* by

$$\varphi^c(y) = \inf_{x \in X} [c(x, y) - \varphi(x)].$$

Similar definitions of c-concavity, c-superdifferential, c-transform can be given for functions $\psi : Y \to \mathbb{R} \cup \{-\infty\}$ by exchanging the roles of X and Y, while keeping the function $c(x, y)$ unchanged.

Remarks 2.34. (i) If X and Y are topological spaces and c is continuous, then a c-concave function is automatically upper semi-continuous on its domain.

(ii) The definition of c-concavity is somewhat obscure; in practice, by applying it, it is essentially impossible to check that a given function is c-concave, even for simple costs. Later we shall give some simpler necessary conditions for c-concavity, in particular cases.

(iii) When $X = Y = \mathbb{R}^n$ and $c(x,y) = |x-y|^2/2$, then the c-concavity of φ is equivalent to the concavity of $\varphi - |x|^2/2$, *not to the concavity of* φ. The convex function we are used to in the optimal transportation theorem is $|x|^2/2 - \varphi(x)$. In this case, of course, there is a differential characterization of c-concavity, namely

$$D^2_{\mathcal{D}'}\varphi \leq I_n.$$

To prevent confusion, the reader is encouraged to make a detailed connection between the generalized notions introduced in Definition 2.33, and the usual notions of convexity theory.

With these definitions in hand, we can easily extend part of Theorem 2.12 to a very general setting. These extensions, extracted from [**225**] and [**15**], are given as exercises below.

Exercise 2.35 (Abstract Legendre duality). Let X and Y be two nonempty sets, and let $c(x,y) : X \times Y \to \mathbb{R} \cup \{+\infty\}$ be an arbitrary function. Let $\varphi : X \to \mathbb{R} \cup \{-\infty\}$.

(i) Show that

$$\forall (x,y) \in X \times Y, \qquad \varphi(x) + \varphi^c(y) \leq c(x,y).$$

Show that $\varphi^{cc} \geq \varphi$. Check that $\psi^{ccc} = \psi^c$, and deduce that $\varphi^{cc} = \varphi$ if and only if φ is c-concave.

Hint: Write $\varphi^{ccc}(y)$ as an inf sup inf and bound the last infimum from above, to prove $\varphi^{ccc} \leq \varphi^c$.

(ii) Show that $x \mapsto c(x,y)$ is c-concave, for any $y \in Y$. Show that an infimum of c-concave functions is c-concave. Prove the following abstract Rockafellar's theorem: *any c-cyclically monotone subset of $X \times Y$ can be included in the c-superdifferential of a c-concave function.* A solution can be found in [**233**] or in [**15**]. This theorem will be referred to as **Rüschendorf's theorem**.

Exercise 2.36 (The dual Kantorovich problem admits a maximizer). Let X and Y be two Polish spaces, and let $\mu \in P(X)$, $\nu \in P(Y)$ be two Borel probability measures. Let $c(x,y)$ be a lower semi-continuous function

defined on $X \times Y$, with values in $\mathbb{R} \cup \{+\infty\}$. Recall from Chapter 1 that the dual Monge-Kantorovich problem consists in maximizing

$$J(\varphi, \psi) = \int_X \varphi \, d\mu + \int_Y \psi \, d\nu$$

over the set Φ_c of all measurable functions $(\varphi, \psi) \in L^1(d\mu) \times L^1(d\nu)$ satisfying the inequality

$$\varphi(x) + \psi(y) \leq c(x, y) \tag{2.61}$$

for $d\mu$-almost all x and $d\nu$-almost all y. If we allow φ, ψ to take values in $\mathbb{R} \cup \{-\infty\}$, it is always possible (redefining φ, ψ on zero-measure sets if necessary) to assume that (2.61) holds for all x and y.

Assume that there exist nonnegative measurable functions $c_X \in L^1(d\mu)$, $c_Y \in L^1(d\nu)$ such that

$$\forall (x, y) \in X \times Y, \qquad c(x, y) \leq c_X(x) + c_Y(y). \tag{2.62}$$

By adapting the second proof of Theorem 2.9, prove that the dual Monge-Kantorovich problem admits a maximizer in the form of a pair of conjugate c-concave functions.

Exercise 2.37 (Optimal transference plans are characterized by their support). We make the same assumptions as in the previous exercise. Let (φ, ψ) be any maximizer in the dual Kantorovich problem. Show that $\pi \in \Pi(\mu, \nu)$ is a minimizer in the primal Kantorovich problem *if and only if* π is concentrated on the set

$$\Big\{ (x, y) \in X \times Y; \qquad \varphi(x) + \psi(y) = c(x, y) \Big\}.$$

Hint: Recall again the proof of Theorem 2.12.

Exercise 2.38 (Cyclical monotonicity via duality). Here we shall see (following [**15**]) that c-cyclical monotonicity of optimal transference plans can be seen as a consequence of the Kantorovich duality. We note that an alternative proof, closer to the arguments sketched in Section 2.3, can be found (under different assumptions) in Gangbo and McCann [**151**].

We consider again Polish spaces X and Y equipped with Borel probability measures μ and ν respectively. We consider a nonnegative, lower semi-continuous cost function $c(x, y)$ with values in $\mathbb{R} \cup \{+\infty\}$, and we assume that $\inf_{\pi \in \Pi(\mu,\nu)} I[\pi] < +\infty$. We denote by π an optimal transference plan in the Kantorovich primal problem.

(i) Assume that the dual Kantorovich problem admits a maximizing pair (φ, ψ) made of conjugate c-concave functions. Let $n \geq 1$, and let σ be a

permutation of $\{1, \ldots, n\}$. Show that

$$\sum_{i=1}^{n} c(x_i, y_{\sigma(i)}) \geq \sum_{i=1}^{n} [\varphi(x_i) + \psi(y_i)]$$

for all $x_1, \ldots, x_n$, $y_1, \ldots, y_n$. Using the Kantorovich duality, show that

$$\varphi(x) + \psi(y) = c(x, y) \qquad \text{for } d\pi\text{-almost all } (x, y).$$

Combine both statements to prove that π is concentrated on a c-monotone subset of $X \times Y$.

(ii) Now we remove the assumption that the dual Kantorovich problem admits a maximizing pair. For this, introduce a maximizing sequence (φ_n, ψ_n) of the dual problem, and show that the nonnegative function $c(x, y) - \varphi_n(x) - \psi_n(y)$ converges to 0 in $L^1(d\pi)$. Deduce that there exist a subsequence n_k and a set Z such that $\pi[Z] = 1$ and $c(x, y) - \varphi_{n_k}(x) - \psi_{n_k}(y)$ converges to 0 on Z. Use this to show that π is concentrated on a c-monotone subset of $X \times Y$.

Remark 2.39. Ambrosio and Pratelli [**15**] proved the following. Assume that c is real-valued, π is concentrated on a c-monotone subset of $X \times Y$, and

$$\mu\Big[x \in X;\ \Big\{\int_Y c(x, y)\, d\nu(y) < +\infty\Big\}\Big] > 0,$$

$$\nu\Big[y \in Y;\ \Big\{\int_X c(x, y)\, d\mu(x) < +\infty\Big\}\Big] > 0.$$

Then π is optimal in the Kantorovich minimization problem, and there exists a maximizing pair in the dual Kantorovich problem. The proof is subtle. A counterexample is also presented in [**15**], in which these conditions are not fulfilled, and there exists at least a transference plan with c-monotone support, which is not optimal. Thus it happens, at least for certain cost functions, that c-cyclical monotone support does not imply optimality.

Remark 2.40 (Duality via cyclical monotonicity). As we learnt from Ambrosio, it is also possible to recover the Kantorovich duality theorem (Theorem 1.3), in full generality, via the method of cyclical monotonicity. The scheme is the following:

1) By approximation, reduce to the case when c is bounded.

2) Let π be optimal in the Kantorovich minimization problem, and let Γ be its support. From Rüschendorf's theorem we know that there exists a c-concave φ such that Γ is included in the c-superdifferential of φ; in particular,

$$(x, y) \in \Gamma \quad \Longrightarrow \quad \varphi(x) + \varphi^c(y) = c(x, y).$$

3) Using the assumption of boundedness of c, one can prove that $(\varphi, \varphi^c) \in L^1(d\mu) \times L^1(d\nu)$. This step requires a little bit of work to estimate φ and φ^c from below.

4) Then

$$\int c(x,y)\, d\pi(x,y) = \int [\varphi(x) + \varphi^c(y)]\, d\pi(x,y) = \int \varphi\, d\mu + \int \varphi^c\, d\nu,$$

which proves the duality.

2.4.2. Differentiable setting. So far we have seen that soft theorems can be proven with the help of abstract convexity notions, with a lot of generality. But now, we would like to have more detailed information when considering a problem in, say, $\mathbb{R}^n$, with a particular cost function, such as a power of the Euclidean distance. And we would like to know whether the *Monge problem* can be solved.

To begin with, let us search for an Euler-Lagrange equation associated with the solution of the Monge minimization problem. If we assume that $\pi = (\mathrm{Id} \times T)\#\mu$ is optimal in the Kantorovich problem, then a continuous variant of the argument sketched in the proof of Proposition 2.24 formally yields the identity

$$\nabla_x c(x, T(x)) = \nabla\varphi(x) \tag{2.63}$$

for some function φ. This calculation can be found in [**134**, Section 8]. By playing a little bit with the dual problem, or by analogy with the case of a quadratic cost function (why?), we suspect that φ in (2.63) might be c-concave, and this suggests that one should study the differentiability properties of c-concave functions. Also, identity (2.63) suggests that *the Monge problem can be uniquely solved when the equation $\nabla_x c(x,y) = F(x)$ can be uniquely solved for $y = y(x)$*. The latter property of the cost c is of paramount importance for the solution of optimal transportation; in the related context of the theory of dynamical systems, it is often called the *twist condition*; much more information can be found in [**267**].

There are at least two situations of interest in which this condition is satisfied:

$$c(x,y) = c(x-y), \qquad c \text{ strictly convex on } \mathbb{R}^n,$$
$$c(x,y) = c(|x-y|), \qquad c \text{ strictly concave on } \mathbb{R}_+.$$

We shall see in the next two subsections that in the first case, eventually the Monge problem can be solved; while in the second case, it can be solved if the supports of μ and ν are disjoint.

2.4.3. The strictly convex case. We start with an exercise, illustrating the utility of the assumption of invertibility of ∇c.

Exercise 2.41 (Connecting c-superdifferential and differential). Let $c(x,y) = c(x-y)$, where c is strictly convex and ∇c invertible on the whole of $\mathbb{R}^n$, with inverse ∇c^* (the $*$ operation being the usual Legendre transform). Let φ be a c-concave function, and x a point at which φ is differentiable. Show that $\partial^c\varphi(x)$ is restricted to just one element:

$$\partial^c\varphi(x) = \big\{x - \nabla c^*(\nabla\varphi(x))\big\}.$$

Strict convexity in itself is not enough to guarantee the invertibility of ∇c on the whole of $\mathbb{R}^n$. As we mentioned in subsection 2.1.3, a sufficient condition for invertibility is that, in addition to strict convexity, c satisfy a *superlinearity* assumption:

$$\lim_{|z|\to\infty} \frac{c(|z|)}{|z|} = +\infty. \tag{2.64}$$

Then the Legendre transform of c is well-defined on the whole of $\mathbb{R}^n$. Indeed, whenever $y \in \mathbb{R}^n$ is fixed, identity (2.64) implies that $x \cdot y - c(x) \le 0$ for x large enough, so that $\sup_x[x \cdot y - c(x)]$ is finite. By an argument similar to the proof of Theorem 2.12, one easily establishes that for almost all x, y,

$$\nabla c^*(\nabla c(x)) = x, \qquad \nabla c(\nabla c^*(y)) = y. \tag{2.65}$$

For technical reasons, Gangbo and McCann [**151**] introduced an additional condition on the cost function c:

(2.66)
Given $r > 0$ and $\theta \in (0,\pi)$, whenever $p \in \mathbb{R}^n$ is far enough from the origin, there should exist a direction $z \in \mathbb{R}^n$ such that, on the truncated cone $K(p,z,\theta)$ with angle $\theta/2$, vertex p, and direction z, defined by

$$K(p,z,\theta) \equiv \Big\{x \in \mathbb{R}^n, \quad |x-p||z|\cos(\theta/2) \le \langle z, x-p\rangle \le r|z|\Big\},$$

c assumes its maximum at p.

This assumption is rather weak, and can easily be checked when c is radial. In the sequel, by abuse of language, when we speak of a "strictly convex, superlinear cost function", we also implicitly assume condition (2.66).

Next, let us say a few words about regularity properties of c-concave functions. As a general rule, they depend on the regularity of the cost c itself. A particularly useful notion in this context is that of **semi-concavity**. By definition, a function φ defined on a convex set of $\mathbb{R}^n$ is said to be semi-concave if $-\varphi$ is semi-convex, recall subsection 2.1.3, 10). One of the main advantages of this definition is that semi-concave functions enjoy all the

regularity properties of convex functions (in particular, existence of a second derivative almost everywhere). Here is a more precise statement.

Lemma 2.42 (Regularity of semi-concave functions). *Let u be a real-valued function defined on a convex subset Ω of $\mathbb{R}^n$, and let $\lambda \in \mathbb{R}$. Then, the following assertions are equivalent:*

(i) $x \longmapsto u(x) - \lambda|x|^2/2$ is a concave function,

(ii) $D^2_{\mathcal{D}'}u \leq \lambda I_n$;

and they imply

(iii) u is twice differentiable in the Aleksandrov sense, and $D^2_{\mathrm{A}}u \leq \lambda I_n$ almost everywhere in Ω.

Conversely, if u is semi-concave and satisfies (iii), then (i) holds true.

The last part of this lemma follows from the fact that the singular part of D^2u can only be a nonpositive measure; so the constant λ which is admissible in (iii) should also be admissible in (i). The rest of the lemma is almost obvious by reduction to the properties of convex functions.

Semi-concavity is sometimes very useful in the theory of partial differential equations, as can be seen in [**133**]. The following proposition from [**151**] illustrates the use of this notion in the present context. By definition, a function is said to lie in $C^{1,1}_{\mathrm{loc}}(\mathbb{R}^n)$ if it is differentiable and its gradient is locally Lipschitz.

Proposition 2.43 (Local regularity of c-concave functions). *Let c be a strictly convex, superlinear cost function, and let φ be a c-concave function. Then, φ is locally Lipschitz on* $\mathrm{Int}(\mathrm{Dom}(\varphi))$. *Moreover, if $c \in C^{1,1}_{\mathrm{loc}}(\mathbb{R}^n)$, then φ is semi-concave.*

Note that the $C^{1,1}_{\mathrm{loc}}$ condition on c excludes all cost functions of the form $|x-y|^p$ for $p < 2$.

After these preparations, the following theorem by Gangbo and McCann [**151**] should appear natural.

Theorem 2.44 (Optimal transportation theorem for a strictly convex cost). *Let c be a strictly convex, superlinear cost on $\mathbb{R}^n$, and let μ, ν be probability measures on $\mathbb{R}^n$, such that the total transportation cost from μ to ν is not always infinite. Assume moreover that μ is absolutely continuous with respect to the Lebesgue measure. Then, there exists a unique optimal transference plan for the Monge-Kantorovich transportation problem, and it has the form $d\pi(x,y) = d\mu(x)\,\delta[y = T(x)]$, or equivalently*

$$\pi = (\mathrm{Id} \times T)\#\mu,$$

where T is uniquely determined $d\mu$-almost everywhere by the requirements that $T\#\mu = \nu$, and

$$T(x) = x - \nabla c^*(\nabla\varphi(x)) \tag{2.67}$$

for some c-concave function φ.

Moreover, if $c \in C^{1,1}_{\mathrm{loc}}(\mathbb{R}^n)$, then the same conclusion holds under the weaker assumption that μ does not give mass to small sets.

2.4.4. The strictly concave case. In the previous subsection, the assumption of strict convexity was important to ensure the invertibility of ∇c. There is another natural case in which this invertibility is guaranteed: when the cost function is concave. Of course this is an abuse of language: we cannot consider a concave cost function, since it would be trivial (a concave, nonnegative function is necessarily constant). But it is possible, and relevant for some applications, to consider cost functions of the form

$$c(x,y) = c(|x-y|), \qquad c(0) = 0,$$

where $c : \mathbb{R}_+ \to \mathbb{R}_+$ is a strictly concave function. By abuse of language, we shall say that c is a strictly concave cost. Such functions are automatically increasing as a function of the distance; and they are quite natural for problems in economics, as emphasized in [**151**].

For strictly concave cost functions, one can prove essentially the same results as for strictly convex costs, with however two major differences. The first is that *all the mass which is shared between μ and ν has to stay in place.* This is not very surprising: a concave cost function defines a distance, and we know from Corollary 1.16 that in such a case, all the mass which is shared by the probability measures can stay in place. The new feature is that for a strictly concave cost function, this mass *has* to stay in place. Note that this property allows one to reduce to the case when μ and ν have disjoint supports.

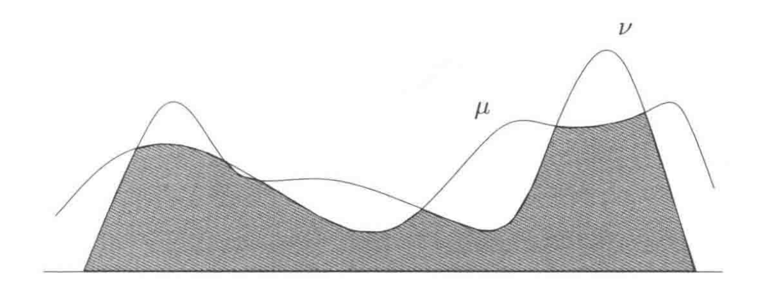


Figure 2.9. Subtraction of the shared mass

The second major difference between the strictly convex and strictly concave cases is that the optimal mapping has a *tendency to reverse the orientation.* The interpretation in terms of economics is that "one long trip is better than several short ones", and "one long trip and one short trip are better than two medium trips". This tendency has been studied qualitatively in McCann [**205**].

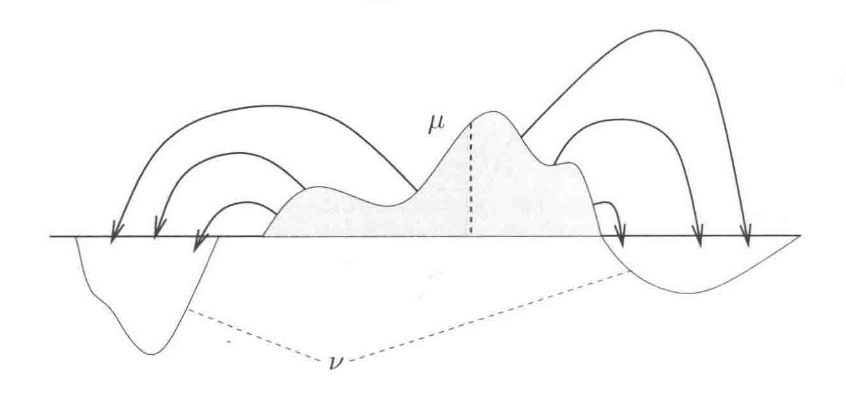


Figure 2.10. Reversal of the orientation

Let us now give a precise result, taken from [**151**].

Theorem 2.45 (Optimal transportation theorem for a strictly concave cost). *Let c be a strictly concave cost on $\mathbb{R}^n$, and let μ, ν be two probability measures on $\mathbb{R}^n$, such that the total transportation cost from μ to ν is not always infinite. Further, assume that μ does not give mass to small sets. Then,*

(i) If μ and ν do not share mass, in the sense that μ and ν are singular to each other, then there exists a unique optimal transference plan for the Monge-Kantorovich transportation problem, and it takes the form $d\pi(x,y) = d\mu(x)\,\delta[y = T(x)]$, or equivalently

$$\pi = (\mathrm{Id} \times T)\#\mu,$$

where T is uniquely determined $d\mu$-almost everywhere by the requirements that $T\#\mu = \nu$ and

$$T(x) = x - (\nabla c)^{-1}(\nabla\varphi(x)) \tag{2.68}$$

for some c-concave function φ.

(ii) If μ and ν do share some mass, then there is still a unique solution to the Kantorovich *problem, which is obtained by first reducing to case (i) after deciding that all the shared mass has to stay in place.*

Exercise 2.46 (The Monge problem often has no solution for a concave cost). Let c be a strictly concave cost on $\mathbb{R}^n$. Use part (ii) of Theorem 2.45 to construct two probability measures μ, ν for which there is no solution to the Monge problem.

2.4.5. Riemannian manifolds. Another way of generalizing the results of Theorem 2.12, without leaving a differentiable context, is to consider cost functions defined in terms of the geodesic distance on a smooth complete Riemannian manifold M, for instance $c(x,y) = d(x,y)^2$. Here and in the sequel, d stands for the usual geodesic distance,

$$d(x,y) = \inf\left\{\int_0^1 \|\dot{z}(t)\|\,dt; \quad z(0)=x,\ z(1)=y\right\} \tag{2.69}$$

$$= \sqrt{\inf\left\{T_* \int_0^{T_*} \|\dot{z}(t)\|^2\,dt; \quad z(0)=x,\ z(1)=y\right\}}, \tag{2.70}$$

where $T_* > 0$ is arbitrary, the infimum is taken over all continuous, piecewise C^1 paths $z : [0,T_*] \to M$ connecting x to y, and $\|\dot{z}(t)\|$ stands for the norm of the tangent vector $\dot{z}(t)$, measured in the tangent space $T_{z(t)}M$ at the point $z(t)$. Elementary considerations in Riemannian geometry show that paths z achieving the infimum in (2.70) are geodesic paths satisfying the equation $\nabla_{\dot{z}}\dot{z} = 0$ (no acceleration). Moreover, they are also minimizing paths between x and $z(t)$, for any $t < T_*$.

Geodesic paths are uniquely determined by their starting point and their initial velocity; if the manifold is complete, then they can be continued forever.

The **exponential map** is defined by the formula

$$\exp_x \xi = z(1),$$

where $z(t)_{t\geq 0}$ is the geodesic path starting from the point x, with velocity $\xi \in T_xM$ (see Figure 2.11). Hence exp is defined from TM into M, where TM is the tangent bundle, i.e. the manifold made of the collection of all tangent spaces T_xM. When M is a complete manifold, this map is well-defined on the whole of TM. The exponential map is a convenient way to collect all geodesic paths into a single object. Note that when M is a Euclidean space (or the torus $\mathbb{R}^n/\mathbb{Z}^n$),

$$\exp_x \xi = x + \xi.$$

In addition to the exponential map, a Riemannian structure automatically induces a **volume** measure, which is the generalization of the Lebesgue measure, and a **gradient** operator, defined by the identity

$$DF(x)\cdot\xi = \langle \nabla F(x), \xi\rangle_x.$$

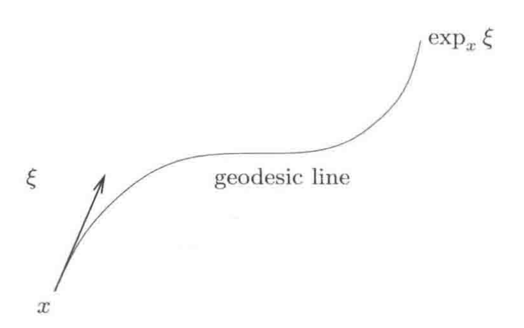


Figure 2.11. The exponential map

The gradient operator maps smooth functions $M \to \mathbb{R}$ into smooth vector fields $M \to TM$. For many more details about all these basic notions of Riemannian geometry, and those which we shall encounter later, we refer for instance to Chavel [**93**] or Do Carmo [**121**].

At the end of the nineties, McCann [**206**] was able to generalize most of Theorem 2.12 to the case of a Riemannian manifold. Below, we state his theorem under slightly simplifying assumptions.

Theorem 2.47 (McCann's theorem). *Let M be a connected, complete smooth Riemannian manifold, equipped with its standard volume measure dx. Let μ, ν be two probability measures on M with compact support, and let the cost function $c(x,y)$ be equal to $d(x,y)^2$, where d is the geodesic distance on M. Further, assume that μ is absolutely continuous with respect to the volume measure on M. Then, the Monge-Kantorovich mass transportation problem between μ and ν admits a unique optimal transference plan, and it has the form $d\pi(x,y) = d\mu(x)\,\delta[y = T(x)]$, or equivalently*

$$\pi = (\mathrm{Id} \times T)\#\mu,$$

where T is uniquely determined, μ-almost everywhere, by the requirements that $T\#\mu = \nu$ and

$$T(x) = \exp_x[-\nabla\varphi(x)] \tag{2.71}$$

for some $d^2/2$-concave function φ.

Remark 2.48. The proof needs the manifold to be of class C^3, i.e. defined by C^3 charts and with a Riemannian metric given by a C^3 function when read in local coordinates.

We note that φ is semi-concave on M, a property which can be defined via local charts; and this implies automatically that φ is differentiable almost everywhere. The key observation in [**206**] is the following: if φ is

differentiable at x, and the function $x' \longmapsto d^2(x', y)/2 - \varphi(x')$ is minimal at $x' = x$, then necessarily $y = \exp_x(-\nabla\varphi(x))$. To prove this, one can show that $u = d^2(\cdot, y)$ is differentiable at x. Being semi-concave, u is automatically superdifferentiable (infinitesimally), i.e. there exists $v \in T_xM$ such that

$$u(\exp_x v) \leq u(x) + \langle v, w \rangle_x + o(|w|). \tag{2.72}$$

But the minimality property will also imply that

$$u(\exp_x v) \geq u(x) + \langle v, \nabla\varphi(x) \rangle_x + o(|w|),$$

and this combined with (2.72) will imply the differentiability of u at x, as desired. With this one can adapt the rest of the Euclidean proof.

2.4.6. The distance case (the original Monge problem). Finally, we discuss a case which was very important for historical reasons, but appears to be extremely degenerate: that of a distance cost function. The cost $c(x, y) = |x - y|$ (the Euclidean distance in $\mathbb{R}^n$) was studied as early as the end of the eighteenth century, by Monge himself [**209**]. Monge considered two-dimensional and three-dimensional problems, in the case when μ and ν are uniform measures on two disjoint sets X and Y; his memoir is still worth reading today, for the beautiful geometrical intuition which is displayed there. He was also able to derive some analytical resolution formulas. His work was considerably improved by Appell [**18**].

Yet, it is only in 1979 that Sudakov [**242**] announced a theorem of existence of solutions to the Monge problem under rather general assumptions. The strategy of Sudakov, which was already used by Monge, and was also used in all subsequent works on the subject, consisted in *reducing to one-dimensional transportation problems.* His proof was based on his theory of "decomposition of measures", which was however later found to be built on fragile grounds, and corrected by Ambrosio [**12**]; below we shall explain the main difficulties in an informal way.

In this situation, formula (2.63) (or the heuristic arguments in [**134**]) suggests that the *direction* of transportation is given by the gradient of some function. But all the information about the distance of transportation seems to be missing.

At the end of the nineties, Evans and Gangbo [**135**] presented an alternative proof of existence for the Monge problem, based on partial differential equations methods which have their own interest. They took advantage of the additional duality described in Section 1.2 of Chapter 1, in particular formula (1.22),

$$\mathcal{T}_d(\mu, \nu) = \inf\Big\{ \|\sigma\|_{L^1}; \quad \nabla \cdot \sigma = \mu - \nu \Big\}. \tag{2.73}$$

This problem can be understood as the limit of a p-Laplace problem as $p \to \infty$, since

$$(2.74) \quad \sup \left[\int \varphi \, d(\mu - \nu) - \int \frac{|\nabla \varphi|^p}{p}; \quad \nabla \varphi \in L^p(\mathbb{R}^n) \right] \\ = \inf \left\{ \frac{\|\sigma\|_{L^{p'}}^{p'}}{p'}; \quad \nabla \cdot \sigma = \mu - \nu \right\}.$$

Assume that μ and ν are absolutely continuous with respect to Lebesgue measure, with respective densities f and g. Then, for given $p > 1$, the maximizer φ_p for the left-hand side of (2.74) (uniquely defined up to an additive constant) is a solution of the **p-Laplace equation**

$$-\nabla \cdot \left(|\nabla \varphi_p|^{p-2} \nabla \varphi_p \right) = f - g.$$

Guided by these considerations, Evans and Gangbo considered the limit of φ_p as $p \to +\infty$. Under the technical assumption that the supports of μ and ν be compact and disjoint, they were able to prove the convergence of φ_p to some Lipschitz function φ_∞ satisfying

$$(2.75) \qquad -\nabla \cdot (a \nabla \varphi_\infty) = g - f, \qquad a \in L^\infty, \quad a \geq 0, \quad |\nabla \varphi_\infty| = 1.$$

This φ_∞ is the **"transportation potential"**: $\nabla \varphi_\infty$ gives the direction of optimal transportation. What is more, the L^∞ function a, which appears as a Lagrange multiplier, gives the *distance* of optimal transportation. Or more precisely (because there is no uniqueness of the optimal transportation), if mass at any point x is transported along the direction $\nabla \varphi_\infty(x)$ by a distance $a(x)$, then the resulting transport is optimal.

This transportation plan can be formalized as follows. By the theory of linear transport equations (see Theorem 5.34), formula (2.75) suggests a natural recipe for transporting μ onto ν: define $T(x)$ as $z(1)$, the solution at time 1 of the ordinary differential equation

$$\begin{cases} z(0) = x, \\ \dot{z}(t) = \dfrac{a(z(t)) \nabla \varphi_\infty(z(t))}{(1-t) f(z(t)) + t g(z(t))}. \end{cases}$$

It is shown in [**135**] that this procedure makes sense and does yield an optimal transportation. In the words of the authors, the proof is "really, really tricky" because of regularity issues. However, a nice feature of this proof is the clear connection with the "∞-Laplace" problem. Many more details, together with many references, can be found in Urbas [**255**].

Remark 2.49. This construction of an optimal transference plan suggests a time-dependent reformulation of the problem. We shall come back to this

in Chapter 5. In this interpretation, the function a should be considered as the *speed* of particles, rather than as a travelled distance.

A much more elementary proof of the existence of an optimal transportation was later obtained by Trudinger and Wang [**254**], and by Caffarelli, Feldman and McCann [**81**]. Also, Ambrosio and Pratelli [**15**] gave an alternative argument, closer in spirit to Sudakov's argument, based on measure disintegration and reduction to one-dimensional problems. All these results assumed some regularity on the norm, like uniform convexity: something like

$$\lambda I_n \leq D^2(\|\cdot\|^2) \leq \Lambda I_n, \qquad \lambda, \Lambda > 0.$$

But in the end, it was yet another approach which allowed Champion and De Pascale [**91, 92**] to establish the existence of a Monge map in full generality:

Theorem 2.50 (Monge's original problem admits a solution). *Let $\|\cdot\|$ be any norm on $\mathbb{R}^n$. Let μ and ν be probability measures on $\mathbb{R}^n$, with finite first order moments:*

$$\int_{\mathbb{R}^n} \|x\| \, d\mu(x) + \int_{\mathbb{R}^n} \|y\| \, d\nu(y) < +\infty.$$

Further assume that μ is absolutely continuous with respect to Lebesgue measure. Then, there exists a solution of Monge's problem, i.e. a measurable map T such that $T\#\mu = \nu$ and T achieves the minimum in Monge's transportation problem with cost $c(x, y) = \|x - y\|$.

Remark 2.51. Here it is important to assume absolute continuity; it would not suffice to require only that μ give no mass to small sets [**15**].

Let us give some informal explanations about the above-mentioned results. The idea in [**81**] is to start from the plain Kantorovich duality, and first find a maximizer of the dual problem

$$\sup \left\{ \int \varphi \, d(\mu - \nu); \qquad \|\varphi\|_{\mathrm{Lip}} \leq 1 \right\} = \mathcal{T}_d(\mu, \nu).$$

With a treatment similar to the one which was applied to the quadratic case in Section 2.1, it is quite easy to show that a transportation mapping T is optimal in the Monge problem if and only if, for $d\mu$-almost every x,

$$\varphi(x) - \varphi(T(x)) = d(x, T(x)). \tag{2.76}$$

This is in fact a particular case of Exercise 2.37 about the characterization of a minimizing measure by its support.

Now, since φ is 1-Lipschitz, formula (2.76) implies that optimal transportation has to take place along the directions defined by $\nabla\varphi$. Once these

lines of transportation, or "**rays**", have been identified, we just have to solve a one-dimensional Monge problem on each of these rays, which is immediate. However, there is a tricky point here! It is not obvious that one can actually perform the *restriction* of μ, ν to these lines, and still get an absolutely continuous measure on each line. This is precisely the point that was missing in Sudakov's argument, and in fact this would be false for an arbitrary family of lines. For instance, Alberti, Kirchheim and Preiss were able to modify a famous example due to Besicovitch, in order to construct a compact family of line segments in $\mathbb{R}^3$ for which the family M of midpoints has strictly positive measure. In that case, the uniform measure μ on M is absolutely continuous, but its restriction to each line segment should be a Dirac mass! In fact, such "pathological" examples are quite well-known in some fields of mathematics, such as the theory of dynamical systems.

But in the case of optimal transportation, the family of rays is not just any family of lines, and one can prove that the restriction of μ, ν is indeed well-defined and absolutely continuous. The technical solution in [**81**] consists in performing a Lipschitz change of variables which locally transforms the family of gradient lines into a "parallel" system. Alternatively, Ambrosio [**12**] noticed that the graph of the map $x \mapsto n(x)$, where $n(x) \in S^{n-1}$ is the direction of the line going through x, is a countable union of Lipschitz graphs, which enables him to use an abstract co-area formula, and arrive at the same conclusion. See [**15**] for detailed explanations and complete lists of references.

As explained before, one peculiarity of this case is that one cannot hope for uniqueness of the transportation map. Recall from Exercise 2.20 that, in dimension one, if the supports of the measures are well separated, *any* transportation is optimal! In several dimensions, uniqueness cannot hold either; however there is not much room for the optimizers, since transport rays are uniquely determined. Indeed, Caffarelli, Feldman and McCann prove the uniqueness of a (belonging to $L^1(\mathbb{R}^n)$ in their setting) and $\nabla\varphi_\infty$ satisfying (2.75), in a sense which is made precise in [**81**].

The optimal existence results from Champion and De Pascale [**92**], on the other hand, use a different strategy which crucially relies on (i) the idea of changing the cost function and using auxiliary minimization probblems, and (ii) a subtle application of Lebesgue's density theory to the optimal transport rays, to establish the convergence of the transport map. We shall not say more about this, and refer to the above-mentioned reference for more information.

Meanwhile, let us say more about the fruitful idea of auxiliary minimization problem. Here comes (in the simplest case of a Euclidean norm)

another uniqueness statement which was suggested by Ambrosio, and proven by Ambrosio and Pratelli [**15**]:

Theorem 2.52 (Selection principle for the Monge problem). *Let μ, ν be probability measures on $\mathbb{R}^n$ with finite first-order moments, and let the cost function $c(x,y)$ be defined as $|x-y|$ (the Euclidean distance). Let $\Pi_*(\mu,\nu)$ stand for the set of all solutions of the Kantorovich problem. Further, assume that μ is absolutely continuous with respect to Lebesgue measure. Then, there is a unique solution of the Monge-Kantorovich problem, in the form $\pi = (\mathrm{Id} \times T)\#\mu$, such that for any strictly convex function $\phi : \mathbb{R}_+ \to \mathbb{R}$, bounded from below, π solves the secondary minimization problem*

$$\inf\left\{\int \phi(|x-y|)\, d\pi(x,y); \qquad \pi \in \Pi_*(\mu,\nu)\right\}. \tag{2.77}$$

Loosely speaking, π in this theorem is obtained by *monotone* transport along each ray.

The following stability result is associated with Theorem 2.52.

Proposition 2.53 (Stability of the Monge solution). *Consider a family of nondecreasing nonnegative strictly convex maps $(\phi_\varepsilon)_{\varepsilon>0}$ on $\mathbb{R}_+$ such that $\phi_\varepsilon(d)$ is convex in ε, converges pointwise to d as $\varepsilon \to 0$, and admits at $\varepsilon = 0$ a right-derivative $\phi(d)$ with respect to ε, where ϕ is strictly convex, real-valued and bounded below. Assume that there exists $\varepsilon_0 > 0$ such that $\int \phi_{\varepsilon_0}(|x-y|)\, d\pi(x,y) < +\infty$ for all $\pi \in \Pi(\mu,\nu)$. Let $(T_\varepsilon)_{\varepsilon>0}$ be solutions of the Monge problem from μ to ν with cost function $\phi_\varepsilon(|x-y|)$. Then, T_ε converges to T in measure as $\varepsilon \to 0$, where T is the optimal mapping in Theorem 2.52.*

Convergence in measure here is to be understood with respect to μ, and means

$$\forall \delta > 0, \qquad \mu\big[|T_\varepsilon - T| > \delta\big] \xrightarrow[\varepsilon\to 0]{} 0.$$

Example 2.54. One natural example of application is for $\phi_\varepsilon(d) = d^{1+\varepsilon}$, which arises when one wants to approximate the Monge-Kantorovich problem with cost function $|x-y|$ by the Monge-Kantorovich problem with cost function $|x-y|^p$, $p > 1$ (an approach initially used by Rachev and Rüschendorf). Then $\phi(d) = d \log d$, which coincidentally is the function defining Boltzmann's entropy. The coincidence is all the more remarkable because entropy criteria are commonly used in the theory of partial differential equations to define uniqueness principles!

The proof of Theorem 2.52 is delicate. Once it is proven, Proposition 2.53 is not too difficult to establish. The arguments in [**15**] rely on the theory of Γ-convergence, but the main idea can be explained in a nutshell, and is

rather instructive, which is why we shall sketch the proof (this is one of the rare results about the distance cost function which does not require too much technical work).

Sketch of proof of Proposition 2.53. We first note that, by convexity, $\phi_\varepsilon(d) \geq d + \varepsilon\phi(d)$. Let π_ε be optimal for the cost ϕ_ε, and let π_0 be a weak cluster point of $(\pi_\varepsilon)_{\varepsilon\to 0}$. We know that

$$\begin{aligned}\int |x-y|\,d\pi_0(x,y) &\leq \limsup_{\varepsilon\to 0} \int |x-y|\,d\pi_\varepsilon(x,y) \\ &\leq \limsup_{\varepsilon\to 0} \int \big[\phi_\varepsilon(|x-y|) - \varepsilon\phi(|x-y|)\big]\,d\pi_\varepsilon(x,y).\end{aligned}$$

Since $-\phi$ is bounded above, and since π_ε is optimal for the cost ϕ_ε, we deduce that for all $\widetilde{\pi} \in \Pi(\mu,\nu)$,

$$\int |x-y|\,d\pi_0(x,y) \leq \limsup_{\varepsilon\to 0} \int \phi_\varepsilon(|x-y|)\,d\widetilde{\pi}(x,y).$$

By using the fact that $\phi_\varepsilon(d)$ is nondecreasing as a function of d, convex as a function of ε, and converges pointwise to d, one can apply the dominated convergence theorem and conclude that the right-hand side converges to

$$\int |x-y|\,d\widetilde{\pi}(x,y)$$

as $\varepsilon \to 0$. Since $\widetilde{\pi}$ is arbitrary, this shows that $\pi_0 \in \Pi_*(\mu,\nu)$, the set of optimal transference plans. Next, let π_* be an arbitrary element of $\Pi_*(\mu,\nu)$. We can write

$$\begin{aligned}\int \phi(|x-y|)\,d\pi_0(x,y) &\leq \limsup_{\varepsilon\to 0} \int \phi(|x-y|)\,d\pi_\varepsilon(x,y) \\ &\leq \limsup_{\varepsilon\to 0} \int \frac{\phi_\varepsilon(|x-y|) - |x-y|}{\varepsilon}\,d\pi_\varepsilon(x,y).\end{aligned}$$

Because π_ε is optimal for the cost ϕ_ε, and $\int |x-y|\,d\pi_\varepsilon \geq \int |x-y|\,d\pi_*$, we can bound this expression above by

$$\limsup_{\varepsilon\to 0} \int \frac{\phi_\varepsilon(|x-y|) - |x-y|}{\varepsilon}\,d\pi_*(x,y) \;=\; \int \phi(|x-y|)\,d\pi_*(x,y),$$

where the equality follows from the fact that $[\varphi_\varepsilon(d) - d]/\varepsilon$ converges monotonically to $\phi(d)$ as $\varepsilon \downarrow 0$. This shows that π_0 is also optimal in the secondary problem (2.77), and therefore should coincide with the optimal plan in Theorem 2.52. Thus the whole family (π_ε) should converge to this optimal plan as $\varepsilon \to 0$. It requires a bit more work to establish the convergence in measure. □

Remark 2.55. As in the case of Theorem 2.50, all the results described in this subsection hold true not only for the Euclidean norm, but also for arbitrary norms $\|\cdot\|$ on $\mathbb{R}^n$ such that $D^2\|\cdot\|^2$ is uniformly bounded from above and below. The treatment of more general norms in $\mathbb{R}^n$ is still an open problem. The tricky point is that, to make sense of the restriction of μ, ν to the transport rays and ensure that the restriction does not have atoms, one uses the regularity of the square norm (either by change of variable as in [**81**], or via the theory of decomposition of measure as in [**12**]). Caffarelli, Feldman and McCann [**81**] also studied the case of the distance cost function on a manifold, or even on a manifold with (convex) holes.

We conclude this section by mentioning some connections with other fields of research. After the work by Evans and Gangbo, there has been some interest in trying to deepen the links between the Monge-Kantorovich problem and the minimal Lipschitz extension problem, formulated by Aronsson in the sixties [**24**]. Aronsson had noticed that a solution of the equation $\langle D^2u \cdot \nabla u, \nabla u\rangle = 0$, set in an open set Ω, with given Lipschitz-continuous boundary data φ, would be an extension of φ to Ω, with minimal Lipschitz norm, satisfying certain interesting properties. This equation is still a popular subject in the theory of viscosity solutions [**103**]. References, and an attempt at connection with the Monge-Kantorovich problem, can be found in Evans [**132**], where the author also discusses some analogies with the Mather-Fathi "weak KAM theory" (see [**137, 138, 131, 139**] and references therein).

2.5. More on c-concave functions

In this section, we dig a little bit in the differential properties of c-concave functions when c is strictly convex.

2.5.1. Differential conditions. Let us look for "simple" necessary conditions for φ to be a c-concave function. In all this subsection, c will be a C^1, strictly convex, superlinear cost function on $\mathbb{R}^n$. Under these assumptions, we know that c-concave functions are locally Lipschitz in the interior of their domain — in particular, almost everywhere differentiable by Rademacher's theorem.

Using Aleksandrov's theorem, we can differentiate the identities (2.65) almost everywhere, to discover that for almost every x, y,

$$D^2c^*(\nabla c(x))\, D^2c(x) = I_n, \qquad D^2c(\nabla c^*(y))\, D^2c^*(y) = I_n. \tag{2.78}$$

We emphasize that these identities a priori only hold almost everywhere: for instance, in the case of a cost function $c(z) = |z|^p$, $p < 2$, then $D^2c(0)$ is not defined, or should be taken to be $+\infty$.

Let us now assume some additional smoothness, namely

$$c \in C^{1,1}_{\mathrm{loc}}(\mathbb{R}^n), \tag{2.79}$$

a condition which excludes the cost $c(z) = |z|^p$ for $p < 2$. Under the assumption (2.79), it is proven in [**151**] that an arbitrary c-concave function φ is also semi-concave. This is actually a consequence of the fact that the cost c itself is semi-concave.

Then let φ be a c-concave function, with c-transform φ^c. For fixed x in $\mathrm{Int}(\mathrm{Dom}(\varphi))$, the mapping

$$\theta_x : y \longmapsto \varphi(x) + \varphi^c(y) - c(x-y)$$

is always nonpositive, with a maximum for $y \in \partial^c\varphi(x)$. (Symetrically, for y fixed, it is maximum for $x \in \partial^c\varphi(y)$.) Using the semi-concavity to differentiate (whenever possible!) this function, we first discover that

$$\nabla\varphi(x) = \nabla c(x-y),$$

in particular $y = x - \nabla c^*(\nabla\varphi(x))$, which means $\{\partial^c\varphi(x)\} = x - \nabla c^*(\nabla\varphi(x))$ for almost all x. Next, expressing the nonposivitity of the second derivative of θ, we find that

$$D^2\varphi(x) \le D^2c(x-y) = D^2c(\nabla c^*(\nabla\varphi)). \tag{2.80}$$

This is a simple *necessary condition* for φ to be a c-concave function.

Moreover, if D^2c is always positive, then, by (2.78), the matrix on the right of (2.80) is nothing else than $[D^2c^*(\nabla\varphi)]^{-1}$. It follows that

$$D^2c^*(\nabla\varphi(x))^{1/2}\, D^2\varphi(x)\, D^2c^*(\nabla\varphi(x))^{1/2} \le I_n. \tag{2.81}$$

Remark 2.56. By a well-known lemma in linear algebra, it also follows that the (a priori non-symmetric!) matrix

$$D^2c^*(\nabla\varphi(x))\, D^2\varphi(x) = \nabla[\nabla c^*(\nabla\varphi)] \tag{2.82}$$

is diagonalizable with real eigenvalues, all of which are bounded above by 1. This remark will be useful later.

Remark 2.57. Contrary to what was written in the first edition of this book, it does not follow from (2.82) that if ψ is c-concave, then

$$\big\langle \nabla c^*(\nabla\psi(x)) - \nabla c^*(\nabla\psi(y)),\, x-y \big\rangle \le |x-y|^2 \tag{2.83}$$

for all $x, y \in \mathbb{R}^n$. However, one can show [**98**] that the most important properties of optimal transportation are preserved in this situation.

Remark 2.58. One may relax the semiconcavity by allowing

$$\varphi(y+h) - \varphi(y) \le \langle p, h\rangle + |h|\,\theta(|h|);$$

then the $C^{1,1}$ regularity may turn into $C^{1,\alpha}$. This is further explored in [**267**, Chapter 10].

It is natural to ask whether $d^2/2$-concavity can be expressed in a differential formulation on a Riemannian manifold M equipped with a geodesic distance d. The following criterion can be established [**100**]: if φ is a $d^2/2$-concave function, then

$$D^2\varphi(x) \leq D^2 \left[\frac{d(\cdot, \exp_x \nabla\varphi(x))^2}{2}\right](x) \tag{2.84}$$

for almost all $x \in M$. Here D^2 is the Hessian operator associated with the manifold M. Note that when $M = \mathbb{R}^n$, inequality (2.84) miraculously reduces to $D^2\varphi \leq I_n$, as expected.

2.5.2. Links with Hamilton-Jacobi equations. Since the cost c is convex, the property of c-concavity can be reformulated in terms of the **time-dependent Hamilton-Jacobi equation**

$$\frac{\partial u}{\partial t} + c^*(\nabla u) = 0.$$

We shall briefly explore this in Chapter 5, and more precisely in subsection 5.4.6.

2.5.3. A regularity estimate. Let us conclude this section with an interesting estimate providing an explicit bound on the size of the derivative of a c-concave function, in terms of a bound for the size of the function itself. Similar estimates are well-known for classical convex functions, see [**136**, p. 236], and the following calculation exemplifies again how many properties of convex functions can be generalized to c-concave functions. The proof is a quantitative adaptation of an argument in [**151**, Appendix B].

Proposition 2.59 (Regularity for c-concave functions again). *Let c be a C^1, strictly convex, superlinear cost function with $c(0) = 0$, and let φ be a c-concave function. Let x be a point in* $\mathrm{Int}(\mathrm{Dom}(\varphi))$ *where φ is differentiable. Assume that φ is bounded on the ball $B = B(x_0, \delta)$ with center x_0 and radius δ. Then, on the ball $B(x_0, \delta/2)$, $|\nabla\varphi|$ is essentially bounded by a number depending only on δ, $\|\varphi\|_{L^\infty(B)}$, and on the growth of c and ∇c at infinity.*

Proof. Let y be in the supergradient of φ at a point $x \in B$. Since $\nabla\varphi(x) = \nabla c(x - y)$, it is sufficient to bound y. For this one writes, for each $z \in B$,

$$\begin{aligned} -\|\varphi\|_{L^\infty(B)} \leq \varphi(z) &\leq \varphi(x) + [c(y-z) - c(y-x)] \\ &\leq \|\varphi\|_{L^\infty(B)} + [c(y-z) - c(y-x)]. \end{aligned}$$

In particular,

$$c(y-x) - c(y-z) \leq 2\|\varphi\|_{L^\infty(B)}.$$

Now, we use the subdifferential inequalities for c:

$$c(y-x) \geq c(y-z) + \nabla c(y-z) \cdot (z-x),$$

to find that

$$\nabla c(y-z) \cdot (z-x) \leq 2\|\varphi\|_{L^\infty(B)}.$$

But

$$0 = c(0) \geq c(y-z) + \nabla c(y-z) \cdot (z-y).$$

Choosing $z \in B$ in such a way that $z-y$ and $z-x$ are collinear, and $|z-x|$ is no less than $\delta/4$, we infer

$$c(y-z) \leq \nabla c(y-z) \cdot (z-x) \frac{|z-y|}{|z-x|} \leq 8\delta^{-1}\|\varphi\|_{L^\infty(B)}|y-z|.$$

In the end, the bound on y follows from the superlinear growth of c. □

Chapter 3

Brenier's Polar Factorization Theorem

In Chapter 2, we saw that gradients of convex functions enjoy remarkable properties in the theory of mass transportation with quadratic cost. More generally, gradients of convex functions constitute a very interesting class of monotone maps in $\mathbb{R}^n$. Brenier's polar factorization theorem [**59**] asserts that any "nondegenerate" vector-valued mapping can be *rearranged* into the gradient of a convex function. In this chapter we shall explain why this theorem is intimately connected with mass transportation, and how it was motivated by problems in fluid mechanics. We shall also explain how it is linked with the standard polar factorization theorem for matrices, and with the Hodge decomposition of vector fields.

The underlying space in all this chapter will be the Euclidean space $\mathbb{R}^n$. For the convenience of readers with a geometrical mind, and also for the sake of intuition, we shall carefully distinguish between vector-valued mappings, which should be thought as mappings from $\mathbb{R}^n$ into $\mathbb{R}^n$, and vector fields, which should be thought of as mappings from $\mathbb{R}^n$ into the tangent bundle $T\mathbb{R}^n$.

3.1. Rearrangements and polar factorization

Definition 3.1 (Rearrangement). Let $m : (W, \lambda) \to (X, \mu)$ be a measurable function between two measure spaces W and X, equipped with measures λ and μ respectively. Another function $\widetilde{m} : (W, \lambda) \to (X, \mu)$ is said to be a rearrangement of m, if the following property holds: whenever $F : X \to \mathbb{R}$ is a measurable function such that $F \circ m \in L^1(d\lambda)$, then

$F \circ \widetilde{m} \in L^1(d\lambda)$ and

$$\int_W (F \circ m)\, d\lambda = \int_W (F \circ \widetilde{m})\, d\lambda. \tag{3.1}$$

Remark 3.2. If $\lambda[W] < +\infty$, then it is equivalent to require that both integrals in (3.1) coincide for any bounded measurable function F. And in any case, it is equivalent to require that (3.1) hold true for all *nonnegative* functions F (why?).

Definition 3.1 means that "one cannot tell the difference between m and $\widetilde{m}$ by looking only at their values". For instance, m and $\widetilde{m}$ should have the same maximum and minimum, but it does not matter at which point. If $X = \mathbb{R}^n$ and $F(x) = |x|^p$, we see from the definition that $\|m\|_{L^p} = \|\widetilde{m}\|_{L^p}$, for any p, so Lebesgue norms are invariant under rearrangement. On the other hand, the great majority of rearrangements of a given smooth function are completely unsmooth; in particular the Sobolev norms $\|\nabla m\|_{L^p}$ are not invariant under rearrangement.

The simplest way to construct rearrangements is via **measure-preserving maps**.

Definition 3.3 (Measure-preserving maps). Let (W, λ) be a given measure space. A measurable function $s : W \to W$ is said to be measure-preserving if

$$s\#\lambda = \lambda.$$

In other words, for any measurable set $A \subset W$, one has $\lambda[s^{-1}(A)] = \lambda[A]$.

Example 3.4. Let Ω be an open subset of $\mathbb{R}^n$, equipped with the n-dimensional Lebesgue measure λ. By the change of variables formula, a C^1 diffeomorphism $s : \Omega \to \Omega$ is measure-preserving if and only if it has unit Jacobian:

$$|\det(\nabla s)| \equiv 1. \tag{3.2}$$

Thus the set of all diffeomorphisms satisfying (3.2) is the group of measure-preserving diffeomorphisms on Ω. An important subgroup of this class is the group of all diffeomorphisms s with $\det(\nabla s) \equiv 1$.

In the sequel, the set of all measure-preserving maps on a measure space (W, λ) will be denoted by $S(W)$. The group of diffeomorphisms $s : \Omega \to \Omega$ with unit Jacobian will be denoted by $SD(\Omega)$, and the subgroup of diffeomorphisms $s : \Omega \to \Omega$ with $\det(\nabla s) \equiv 1$ will be denoted by $G(\Omega)$.

Remark 3.5. Measure-preserving diffeomorphisms constitute but a small part of $S(\Omega)$. For instance, $SD((0,1))$ is reduced to just $\{\mathrm{Id}, -\mathrm{Id}\}$, but

there are many elements in $S((0,1))$ which are not diffeomorphisms, like $s(x) = 2x \,(\mathrm{mod}\, 1)$, $s(x) = 2x1_{x\le 1/2} + (1-2x)1_{x>1/2}$, etc.

Exercise 3.6. Show that $S(\Omega)$ is contained in a sphere of $L^2(\Omega)$ (in particular, it is not convex).

The following elementary proposition provides a link between measure-preserving maps and rearrangements.

Proposition 3.7 (Measure-preserving maps and rearrangements). *Let (W,λ) be a measure space. If $s \in S(W)$ and $\widetilde{m} = m \circ s$, then $\widetilde{m}$ is a rearrangement of m. "Conversely", if $\widetilde{m}$ is a one-to-one rearrangement of m, then $\widetilde{m}^{-1} \circ m \in S(W)$.*

Proof. 1. Assume $\widetilde{m} = m \circ s$, $s \in S(W)$. Then, for all measurable $F : W \to \mathbb{R}_+$,

$$\int (F \circ \widetilde{m})\, d\lambda = \int (F \circ m) \circ s \, d\lambda = \int (F \circ m)\, d(s\#\lambda) = \int (F \circ m)\, d\lambda.$$

So $\widetilde{m}$ is a rearrangement of m.

2. Let $\widetilde{m}$ be a one-to-one rearrangement of m. Define $s = \widetilde{m}^{-1} \circ m$ and consider any measurable nonnegative function F on W. Then

$$\int F\, d(s\#\lambda) = \int (F \circ s)\, d\lambda = \int (F \circ \widetilde{m}^{-1}) \circ m \, d\lambda = \int (F \circ \widetilde{m}^{-1}) \circ \widetilde{m}\, d\lambda = \int F\, d\lambda.$$

So s is measure-preserving. □

So far all is extremely general. To go further, we shall focus on the differentiable case when W is a subset of $\mathbb{R}^n$. A general question is whether there exists a particular class $\mathcal{R}$ of functions, with nice properties, such that any measurable function $m : W \to X$ admits a rearrangement in $\mathcal{R}$. When $W = \mathbb{R}^n$ and $X = \mathbb{R}_+$, there is a very famous such class, namely the set of all functions of the form $F(|x - x_0|)$, where x_0 is arbitrary and F is nonnegative nonincreasing. A user-friendly introduction to this theory of **radially symmetric monotone rearrangements** can be found in [**190**]. Here we shall consider a different situation, in which $W \subset \mathbb{R}^n$ and $X \subset \mathbb{R}^n$, and the class $\mathcal{R}$ will be the set of gradients of convex functions. The main result is the following striking theorem by Brenier. Recall that $|A|$ stands for the Lebesgue measure of A.

Theorem 3.8 (Brenier's polar factorization theorem). *Let Ω be a bounded subset of $\mathbb{R}^n$, with positive Lebesgue measure. Let $h : \Omega \to \mathbb{R}^n$ be*

an L^2 vector-valued mapping satisfying the nondegeneracy condition

$$\text{(3.3)} \qquad \text{for any small set } N \text{ in } \mathbb{R}^n, \qquad |h^{-1}(N)| = 0.$$

Then, there exist a unique rearrangement $\nabla\psi$ of h in the class of L^2 gradients of convex functions, and a unique measure-preserving map $s \in S(\Omega)$, such that

$$h = \nabla\psi \circ s.$$

Moreover, s is the unique L^2 projection of h onto $S(\Omega)$.

Remarks 3.9. (i) The L^2 norms in this statement are taken with respect to Lebesgue measure on Ω. When we say that $\nabla\psi$ is the gradient of a convex function, we mean that $\nabla\psi$ is the restriction to Ω of the gradient of a convex function ψ on $\mathbb{R}^n$. When we say that s is the L^2 projection of h onto $S(\Omega)$, we mean that s minimizes $\|h - \sigma\|_{L^2(\Omega)}$ among all $\sigma \in S(\Omega)$.

(ii) We emphasize that the vector-valued mapping h below should *not* be understood as a tangent vector field, but as a plain mapping. Similarly, $\nabla\psi$ should not be seen as a tangent vector field, see Remark 2.13 (iv).

(iii) The nondegeneracy condition (3.3) means that the image measure of the Lebesgue measure by h does not give mass to small sets; we recall that these are, by convention, measurable sets with Hausdorff dimension at most $n-1$. Of course (3.3) is satisfied if $|h^{-1}(N)| = 0$ for all N with zero measure, which was Brenier's original assumption [**59**]. We further note that (3.3) is not a necessary condition for the existence of the polar factorization; in this respect it has been improved by Burton and Douglas [**70**]. On the other hand, this condition is necessary for uniqueness.

(iv) Even if this is not clear from the statement above, Theorem 3.8 has an intrinsic formulation, in the sense of Riemannian geometry [**206**]. A rather general statement reads as follows: let M be a compact Riemannian manifold, let λ be the normalized volume on M, and let $h : M \to M$ be a measurable map such that $h\#\lambda$ is absolutely continuous; then there exists a unique pair $(\nabla\varphi, s)$ such that $h(x) = \exp_{s(x)} -\nabla\varphi(s(x))$, where s is measure-preserving and φ is $d^2/2$-concave. Moreover, s is the unique solution of the minimization problem

$$\min \left\{ \int_M d(u(x), \sigma(x))^2 \, dx; \quad \sigma \text{ is measure-preserving} \right\}.$$

The polar factorization theorem is very intimately connected with the Monge transportation problem; in fact, in Section 3.3 below, we shall prove a slightly more general result, which is almost equivalent to Theorem 2.12. A most remarkable feature of Brenier's reformulation is the interpretation of s in Theorem 3.8 as a **projection** onto the nonconvex set $S(\Omega)$.

But before discussing the proof and consequences of this result, let us try to give an idea of Brenier's original motivations, completely out of the field of optimal transportation.

3.2. Historical motivations: fluid mechanics

In this section, we shall explain why problems in fluid mechanics naturally led to the search for a projection operator onto the set of measure-preserving maps. We shall be content here with only the most basic considerations, and refer the reader to Brenier's works [**58, 59, 60, 61, 62, 63**] for much more.

3.2.1. The incompressible Euler equation. The Euler equation is one of the most basic and oldest equations in fluid mechanics, going back to the eighteenth century, and yet still rich in mysteries. In its simplest version it models an incompressible, inviscid fluid in a bounded, smooth open set $\Omega \subset \mathbb{R}^n$ ($n = 2$ or 3). The unknown is the velocity field of the fluid,

$$v(t,x) : \mathbb{R}_+ \times \Omega \longrightarrow \mathbb{R}^n \qquad (= \text{tangent space to } \Omega),$$

and the incompressible Euler equation reads

$$(3.4) \qquad \begin{cases} \dfrac{\partial v}{\partial t} + v \cdot \nabla v = -\nabla p, \\[2ex] \nabla \cdot v = 0. \end{cases}$$

Here, by definition, $\nabla \cdot v = \sum_i (\partial v_i / \partial x_i)$ is the divergence of v, and $v \cdot \nabla v = (v \cdot \nabla) v$ is the vector field whose i^{th} component is

$$(v \cdot \nabla) v_i = \sum_{j=1}^{n} v_j \frac{\partial v_i}{\partial x_j}.$$

Equation (3.4) must be supplemented with a **boundary condition**, most typically

$$v \text{ tangent to the boundary of } \Omega.$$

The condition $\nabla \cdot v = 0$ means that the fluid is **incompressible**. As we shall see in the next subsection, it is a mathematical way to express the fact that the volume of an "element of fluid" does not change during the time evolution. A vector field v satisfying this incompressibility condition is said to be **divergence-free**.

The unknown $p(t,x) : \mathbb{R}_+ \times \Omega \to \mathbb{R}$ is the **pressure**. Its evolution is given by no equation, and it does not have to be specified at the initial time for the evolution problem to make sense. In fact, one can see it as a Lagrange multiplier associated with the infinite-dimensional incompressibility constraint.

The **Cauchy problem** for the Euler equation reads as follows: *given a velocity field v_0 on Ω, tangent to the boundary and divergence-free, satisfying certain conditions to be made precise, prove existence and uniqueness of a solution to* (3.4) *satisfying $v(0,\cdot) = v_0$.*

Of course this formulation does not really make sense until one has explained what "solution" means and what properties we should expect from it. The Cauchy problem is considered by most specialists to be satisfactorily solved in dimension $n = 2$ under the condition that $\nabla \wedge v_0$ (curl of v_0) is bounded (this is **Youdovich's theorem**). In dimension $n = 3$ it is an extremely famous open problem. We refer to the books [**200, 94, 194**] for a detailed account on this question.

The following a priori estimate (law of conservation of energy) suggests that a natural function space in which to look for solutions of the Euler equation would be $L^2(\Omega;\mathbb{R}^n)$.

Proposition 3.10 (Energy conservation for smooth solutions of the Euler equation). *Let v be a smooth solution of* (3.4). *Then, the total kinetic energy*

$$\int_\Omega |v(t,x)|^2\,dx = \|v(t,\cdot)\|^2_{L^2(\Omega;\mathbb{R}^n)}$$

is preserved with time.

Sketch of proof. First,

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |v|^2 = \int_\Omega v\cdot\frac{\partial v}{\partial t} = -\int_\Omega v\cdot(v\cdot\nabla v) - \int_\Omega v\cdot\nabla p.$$

But on one hand,

$$\begin{aligned}\int_\Omega v\cdot\nabla p &= -\int_\Omega (\nabla\cdot v)\,p \quad &&\text{[because } v \text{ is tangent to the boundary of } \Omega]\\ &= 0 &&[\text{because } \nabla\cdot v = 0],\end{aligned}$$

and on the other hand,

$$\begin{aligned}\int_\Omega v\cdot(v\cdot\nabla v) &= \sum_{1\le i,j\le n}\int_\Omega v_i v_j \partial_j v_i = \frac{1}{2}\sum_{ij}\int_\Omega v_j\partial_j(v_i^2) = \frac{1}{2}\int_\Omega v\cdot\nabla|v|^2\\ &= -\frac{1}{2}\int_\Omega(\nabla\cdot v)|v|^2 = 0.\end{aligned}$$

□

Remark 3.11. Actually, many people think that, in dimension $n = 3$, solutions of (3.4) should not in general have enough regularity for the conclusion of this proposition to hold, so that the energy could be decreasing. This was first suggested by the great physicist Onsager [**212**].

3.2.2. Lagrangian formulation. The point of view used in the last subsection, in which the unknown was a time-dependent velocity field, is called the **Eulerian** formulation. There is an alternative, formally equivalent, way of looking at fluid mechanics: the **Lagrangian** point of view, which focuses on the trajectories of particles. Let us profit by this occasion to recall the most basic facts concerning these two approaches, which will later turn out to be very important in the study of the time-dependent Monge-Kantorovich problem; more can be found in [**110**, Chapter II]. By the way, we note that the "Lagrangian" point of view was apparently introduced by Euler, while the "Eulerian" point of view should be attributed to Bernoulli and D'Alembert.

• In an Eulerian description, one stares at a given, fixed point of space x, and measures the velocity $v(t,x)$ of fluid particles going through this point at time t.

• In a Lagrangian description, one puts a label on each particle, and then studies the trajectory of each labelled particle. For instance, assuming that we label particles according to their initial position x_0, we denote by

$$x = m(t, x_0)$$

the position at time t of a particle that was located at position x_0 at time 0. It is usually assumed that for each time t, the map $x_0 \longmapsto m(t,x_0)$, defined on Ω, is one-to-one.

To switch between these two descriptions, it suffices to use the identities

$$\begin{cases} v\big(t, m(t,x_0)\big) = \dfrac{d}{dt}\, m(t,x_0), \\ m(0,x_0) = x_0. \end{cases} \tag{3.5}$$

It is important to keep in mind the Eulerian expression of the **Lagrangian acceleration**: by differentiating (3.5) with respect to time, one finds that

$$\frac{d^2}{dt^2}\, m(t,x_0) = \left[\frac{\partial v}{\partial t} + (v\cdot\nabla v)\right]\big(t, m(t,x_0)\big), \tag{3.6}$$

which explains the occurrence of the convective derivative $v\cdot\nabla v$ in the Euler equation.

One immediately sees that if two particles coming from different initial locations occupy the same position at some time t (i.e. $m(t,x_1) = m(t,x_2)$), then there is an ambiguity to remove in the definition of the velocity field (3.5). This is one possible reason for imposing injectivity of the map m (other reasons would rest on physical interpretation). Moreover, in the context of the incompressible Euler equation, it is natural to require that the map m also be surjective: if not, there would be some vacuum created

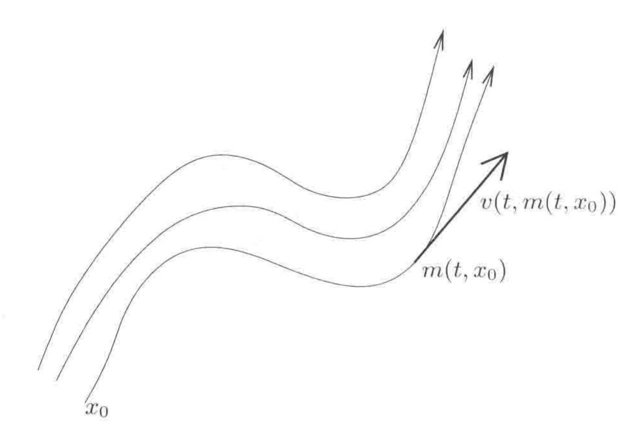


Figure 3.1. Eulerian vs. Lagrangian description

inside the domain, contradicting the fact that the fluid has constant density (incompressibility). This is why it seems natural to search for $(m(t,\cdot))_{t\geq 0}$ as a family of **diffeomorphisms** from Ω to Ω.

The incompressibility constraint can be recast in terms of m: if v is regular enough (say C^1), then

$$\nabla \cdot v = 0 \Longleftrightarrow \det\left(\frac{\partial m}{\partial x_0}\right) \equiv 1. \tag{3.7}$$

Indeed, the identity on the right-hand side of (3.7) is obviously satisfied at time 0, since $m(0,\cdot)$ is the identity map ($m(0,x_0) = x_0$); then (3.7) is a consequence of the identity

$$\frac{\partial}{\partial t} \log\det\left[\frac{\partial m}{\partial x_0}\right] = (\nabla \cdot v)\big(t, m(t,x_0)\big). \tag{3.8}$$

The proof of (3.8) is left as an exercise; we recall that the differential of the determinant application, taken at the matrix A, is given by

$$D\det\Big|_A \cdot H = (\det A)\ \mathrm{tr}\,(A^{-1}H) = \mathrm{tr}\,\big((\mathrm{com}\ A)^T H\big). \tag{3.9}$$

Here com A stands for the comatrix, or matrix of cofactors of A, and only the second expression makes sense if A is not invertible. To prove (3.9), one can reduce to the easy case $A = I_n$ by means of the formula $\det(AM) = (\det A)(\det M)$.

Let us recapitulate the above informal considerations: in Lagrangian formulation, the Euler equation becomes an evolution equation for a map $t \mapsto m(t,\cdot)$, with values in *the group* $G(\Omega)$ *of diffeomorphisms* $\Omega \to \Omega$ *with unit determinant.* To recall this, we shall use the letter g for the trajectory map m. In particular, g is *measure-preserving*: it pushes Lebesgue measure

(restricted to Ω) forward to itself. The physical interpretation is that the volume of a set of particles is kept constant under time-evolution, which is precisely the *incompressibility*.

Thus, we rewrite (3.5) as

$$v\big(t, g(t, x_0)\big) = \frac{d}{dt}\, g(t, x_0), \qquad \text{or} \quad v = \frac{\partial g}{\partial t} \circ g^{-1}. \tag{3.10}$$

By (3.6), the Euler equation translates into an equation on the trajectory field $t \longmapsto g(t, \cdot)$ of $\mathbb{R}_+$ into $G(\Omega)$,

$$\frac{d^2}{dt^2}\, g(t, x_0) = -\nabla p\big(t, g(t, x_0)\big). \tag{3.11}$$

3.2.3. Arnold's interpretation. Starting from the seminal paper [**21**] of Arnold, the following formal interpretation of the Euler equation was developed and exploited by Arnold and coworkers, see in particular [**22**]:

The Euler equation is the equation of geodesics on $G(\Omega)$, endowed with the Riemannian structure inherited from the Euclidean space $L^2(\Omega; \mathbb{R}^n)$.

Remarks 3.12. (i) Recall from Exercise 3.6 that $G(\Omega)$ is included in a sphere of $L^2(\Omega; \mathbb{R}^n)$.

(ii) The only reason why we use $L^2(\Omega; \mathbb{R}^n)$ in place of $L^2(\Omega; \Omega)$ is that the latter is not a vector space.

Let us explain Arnold's interpretation. The following discussion is purely formal. Recall that a geodesic on a Riemannian manifold M is a path $\gamma(t)$ which minimizes the distance

$$\sqrt{\int_{t_1}^{t_2} |\dot{g}(t)|^2\, dt} \tag{3.12}$$

among all curves $g : [t_1, t_2] \to M$ constrained by the boundary conditions $g(t_1) = \gamma(t_1)$, $g(t_2) = \gamma(t_2)$, and this minimization property should hold true whenever t_2 is close enough to t_1. It is equivalent to stating that the acceleration of the curve γ, viewed from the tangent space to the manifold, vanishes identically. Here, in Arnold's interpretation, we consider the Riemannian structure on $G(\Omega)$ inherited from $L^2(\Omega)$, and this simply means that the acceleration d^2g/dt^2 should be orthogonal to the tangent space $T_{g(t)}G(\Omega)$ in $L^2(\Omega; \mathbb{R}^n)$.

So let us determine this tangent space. Recall from the preceding discussion that a path $g(t)$, starting from $g_0 \in G$, stays in G if and only if $\partial g/\partial t$ is tangent to the boundary, and

$$\nabla \cdot \left[\frac{\partial g}{\partial t} \circ g^{-1}\right] = 0.$$

Thus, tangent vectors in T_gG are all vector fields h such that $\nabla\cdot(h\circ g^{-1}) = 0$, or equivalently $h = w_0 \circ g$, where w_0 lies in D_0, the space of divergence-free vector fields. Using the fact that g is a measure-preserving diffeomorphism, one immediately checks that $(T_gG)^\perp$ is the space of all vector fields $q_0 \circ g$, where $q_0 \in D_0^\perp$, and $D_0^\perp$ is the orthogonal subspace to D_0 in L^2. Now, it is an easy consequence of the Helmoltz decomposition (see subsection 3.4.2) that, under reasonable regularity conditions on Ω,

$$D_0^\perp = \big\{-\nabla p, \quad p : \Omega \to \mathbb{R}\big\}.$$

So the equation for geodesics becomes

$$\frac{d^2}{dt^2}\, g(t) = -\nabla p(t, g(t)).$$

This is exactly the incompressible Euler equation (3.11), in Lagrangian formulation.

Note that in this picture, the integral which appears in (3.12) is interpreted (up to a factor $1/2$ appearing in the definition of kinetic energy) as the **action** of the trajectory $g(t,x)$, i.e. the integral over time of the kinetic energy associated with the velocity field.

Even though these considerations are formal, on some occasions they do give precious hints, see [**22**]. As we just saw, they also provide an interpretation of the pressure field. In fact, to put the discussion above on a more sound basis, it would be sufficient to have suitable regularity estimates on this pressure field. For instance one can show that, if $g(t, x_0)$ is a smooth solution of the Euler equation, with a pressure field p bounded in $C^2(\Omega)$, uniformly in time, then there exists $\varepsilon > 0$ such that for $|t_1 - t_2| < \varepsilon$,

$$\int_{t_1}^{t_2} \left(\int_\Omega \left| \frac{\partial g}{\partial t}(t, x_0) \right|^2 dx_0 \right) dt \le \int_{t_1}^{t_2} \left(\int_\Omega \left| \frac{\partial \gamma}{\partial t}(t, x_0) \right|^2 dx_0 \right) dt$$

for any other trajectory mapping γ with $\gamma(t_1) = g(t_1)$ and $\gamma(t_2) = g(t_2)$. If Ω is convex, one can choose $\varepsilon = \pi/\sqrt{\|D^2 p\|_{L^\infty}}$, see [**61**].

On the other hand, such regularity estimates constitute a formidable open problem. If one wants to prove useful statements about the solutions of the incompressible Euler equation with the help of the above formalism, it is a priori impossible to apply the general tools of Riemannian geometry, because G is infinite-dimensional, and very singular in some sense.

The following partial result has been known for a long time [**129**]: if g_0 and g_1 are two smooth elements of $G(\Omega)$, close enough in a Sobolev norm of high enough order, then there exists at least one geodesic path in $G(\Omega)$ connecting g_0 and g_1. But in some circumstances, geodesics may be nonunique (which is rather common), or may simply not exist (which is

more frightening!). Indeed, Shnirelman [238] managed to construct a diffeomorphism $h \in G([0,1]^3)$ such that there is no geodesic path connecting the identity mapping to h. Very briefly, here is an idea of the argument: the diffeomorphism h really is two-dimensional (it keeps the vertical coordinate, z, unchanged). Then one can show that a minimizing path connecting Id to h would have to use the third dimension, because two-dimensional fluid movements are much more constrained than three-dimensional. But if such a three-dimensional minimizing path $(h_t)_{0 \le t \le 1}$ existed, then one could construct a strictly better one by setting $\widetilde{h}_t(x,y,z) = h_t(x,y,2z)$ for $0 < z < 1/2$, $\widetilde{h}_t(x,y,z) = h_t(x,y,2z-1)$ for $1/2 < z < 1$. After a little bit of smoothing on the interface $z = 1/2$, this doubling procedure yields a path $\widetilde{h}$ which still connects Id with h, but has less kinetic energy in the vertical direction (why?), and a contradiction results. See [238] for the implementation, and various insights into this problem.

Remark 3.13. Shnirelman's proof was actually inspired by a historically famous counterexample opposed to Riemann's sloppy proof of what is now called the Riemann mapping theorem. The counterexample was intended to demonstrate that "not all variational problems have a solution", and can be stated as follows. Consider the set of all curves $u : [0,1] \to \mathbb{R}$ with $u(0) = u(1) = 0$, having vertical tangents at 0 and 1. Among these curves, there is no minimizer of the length functional $L(u) = \int_0^1 \sqrt{1 + u'(x)^2}\, dx$. As an easy exercise, the reader can prove this assertion with a reasoning similar to the one sketched above.

Among other results, Shnirelman proved that when Ω is a *three-dimensional* domain ($n = 3$), the diameter of $G(\Omega)$ is finite [**238, 239**], while this is false for a two-dimensional domain, due to the symplectic structure of $G(\Omega)$. This is another indication that incompressible two-dimensional flows are much more constrained than three-dimensional ones.

3.2.4. The problem of approximate geodesics. Motivated by these ideas, Brenier endeavored to study generalized notions of geodesics. For their construction he developed the notion of **approximate geodesics** on $G(\Omega)$, as an approximation of incompressible Euler equations. Here is the very simplest example: let g_0, g_1 be two given elements of $G(\Omega)$; then the problem is to find $g_{1/2} \in G(\Omega)$, achieving the minimum of

$$\mathcal{A}(g) = \|g - g_0\|_{L^2}^2 + \|g_1 - g\|_{L^2}^2 = \frac{1}{2}\|g_0 - g_1\|_{L^2}^2 + 2\left\| g - \frac{g_0 + g_1}{2} \right\|_{L^2}^2$$

among all $g \in G(\Omega)$. To solve this problem, $g_{1/2}$ should be exactly the L^2 projection of $h = (g_0 + g_1)/2$ onto $G(\Omega)$. Even if this construction is very

naive, it can be refined by allowing more intermediate points, and in the end it leads to the construction of approximate geodesics on $G(\Omega)$.

But exactly how do we know that the L^2 projection of a given $h \in L^2(\Omega)$ onto $G(\Omega)$ is well-defined? A well-known theorem states that the L^2 projection on a closed, convex subset of L^2 is well-defined. But recall that $G(\Omega)$ is not convex; and it is not closed either! Actually, the following can be shown: whenever Ω is a smooth subset of $\mathbb{R}^n$, $n \geq 2$, *any measure-preserving map on Ω can be approximated, in L^2 norm, by a sequence of diffeomorphisms with unit Jacobian.* For instance, even a map s with $\det(\nabla s) \equiv -1$ may be approximated by elements in $G(\Omega)$. Such results belong to the folklore on the subject; some proofs may be found in the PhD thesis of Roesch [**229**], or in a paper [**66**] by Brenier and Gangbo.

To continue this program, it was therefore natural to relax the projection problem, and replace $G(\Omega)$ by its closure $S(\Omega)$. Then Brenier's polar factorization theorem provides sufficient conditions for the projection onto $S(\Omega)$ to be well-defined. This is the historical motivation of the polar factorization theorem!

As announced earlier, we shall not develop more on this line, and refer to Brenier's works [**58, 59, 61, 60, 62, 63**] for more. In [**62**] Brenier discussed the possibility of constructing geodesic paths in a generalized sense, allowing particles to split along various trajectories — in the same way that in Kantorovich's formulation of the mass transportation problem, one allows mass to be split. Of course, under this generalization the property of the density of particles being constant in space and time (homogeneity) is lost, and the equation $\nabla \cdot v = 0$ should be replaced by

$$\frac{\partial \rho}{\partial t} + \nabla_x \cdot (\rho v) = 0, \tag{3.13}$$

where $\rho(t, x)$ stands for the density of particles; see Theorem 5.34 below for explanations. Similarly, the incompressible Euler equation (3.4) should be replaced by

$$\frac{\partial \rho}{\partial t} + \nabla_x \cdot (\rho v \otimes v) + \rho \nabla_x p = 0. \tag{3.14}$$

So the objects constructed by Brenier are in fact generalized density functions of the form $\rho(dx_0, t, x)$. The meaning of such a ρ is that it "counts" the quantity of particles initially in the volume element dx_0 around x_0, occupying the position x at time t. In this formalism, a generalized solution ρ reduces to a classical one if ρ is a Dirac mass with respect to its first argument x_0 (just as the Kantorovich problem reduces to the Monge problem if the transference plan is a Dirac mass in its y argument, formula (7)).

After these preparations, we can quote the main result of [**62**], just to give the reader a more precise idea:

Theorem 3.14 (Existence of generalized geodesics in $S(\Omega)$). *Let $\Omega = [0,1]^3$ and let $T > 0$. Let $h \in S(\Omega)$ be a measure-preserving map; this will be the target. Assume that it really is two-dimensional, in the sense that*

$$\forall x \in [0,1]^3, \quad h(x_1, x_2, x_3) = (H(x_1, x_2), x_3).$$

Then, there exist time-dependent measures ρ (to be thought of as a density of particles), m (to be thought of as a density of momentum) defined on the extended measure space $\Omega \times ([0,T] \times \Omega)$, satisfying the following properties. First, $m = v\rho$, where v is an $\mathbb{R}^d$-valued L^2 density, to be thought of as a local velocity of particles. Next, $\rho(dx_0; t, x)$ is a probability measure with respect to x_0, for almost all t and x; and it satisfies the generalized incompressible Euler equations (3.13) *and* (3.14) *in the sense of distributions. Furthermore, the density of particles matches with the initial and final distributions:*

$$\rho(dx_0; 0, x) = \delta_{[x_0 = x]}, \qquad \rho(dx_0; T, x) = \delta_{[h(x_0) = x]}.$$

Finally, the total (generalized) action

$$A = \int_0^T \left[\int_{\Omega \times \Omega} \frac{|v(x_0; t, x)|^2}{2} \, \rho(dx_0; t, dx) \right] dt$$

is minimal in the following generalized sense: for all $\eta > 0$ there exists $\varepsilon > 0$ such that if $g(t,x)$ is a solution of the incompressible Euler equation (3.11) *satisfying $\|g(T,\cdot) - h\|_{L^2} \leq \varepsilon$, then*

$$A \leq \left(\int_0^T \frac{1}{2} \int_\Omega |u(t,x)|^2 \, dt \right) + \eta,$$

where u is the velocity field associated with g.

Note that the choice of a two-dimensional measure-preserving map is not so much a simplifying assumption; in fact, from our discussion of Shnirelman's work we understood that it is one of the worst cases for these geodesic problems.

3.3. Proof of Brenier's polar factorization theorem

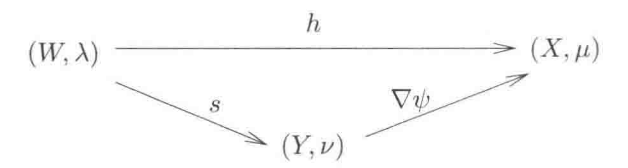
We restate here the polar factorization theorem, in a slightly more general form:

Theorem 3.15 (Brenier's polar factorization theorem again). *Let W and Y be measurable subsets of $\mathbb{R}^n$, and let $\lambda \in P(W)$, $\nu \in P(Y)$, with $\int_Y |y|^2 \, d\nu(y) < +\infty$. Let $h : W \to X \subset \mathbb{R}^n$ be an $L^2(d\lambda)$ mapping, and let*

$\mu = h\#\lambda$. Assume that both μ and ν give no mass to small sets. Then, there exists a unique pair $(\nabla\psi, s)$ such that

$$\begin{cases} \psi : Y \to X \text{ is a convex function,} \\ s : W \to Y \text{ pushes } \lambda \text{ forwards to } \nu \quad (s\#\lambda = \nu), \\ h = \nabla\psi \circ s. \end{cases} \tag{3.15}$$

In particular, the following diagram is commutative:



Moreover, s is the unique projection of h onto $S(W, Y)$, the set of mappings σ such that $\sigma\#\lambda = \nu$.

Remarks 3.16. (i) Again, "convex" means restriction of a convex function; and when we say that $\nabla\psi$ is unique, we mean that it is uniquely determined $d\nu$-almost everywhere on Y.

(ii) Since all elements in $S(W, Y)$ have the same L^2 norm, the last statement of the theorem is equivalent to the statement that s should *maximize* $\langle h, \sigma\rangle_{L^2(\lambda)} = \int h\sigma \, d\lambda$ among all σ's in $S(W, Y)$.

(iii) Theorem 3.8 is just the case $W = Y = \Omega$, equipped with the normalized Lebesgue measure.

Proof of Theorem 3.15. 1. We look for a map s which minimizes

$$\int_W |h(w) - \sigma(w)|^2 \, d\lambda(w)$$

among all $\sigma \in S(W, Y)$. If we introduce the image measure $\pi = (h \times \sigma)\#\lambda$, then our problem becomes

$$\min_\sigma \left\{ \int_{X\times Y} |x - y|^2 \, d\pi(x, y); \quad \pi = (h \times \sigma)\#\lambda; \quad \sigma\#\lambda = \nu \right\}. \tag{3.16}$$

In probabilistic terms, π is the joint law of (h, σ), thought of as a couple of random variables on W. Note that

$$\pi \in \Pi(h\#\lambda, \sigma\#\lambda) = \Pi(\mu, \nu),$$

where $\mu = h\#\lambda$. If we exhibit s such that $s\#\lambda = \nu$ and $s = \nabla\varphi \circ h$, for some convex function φ, then $\pi = (h \times s)\#\lambda$ will be concentrated on the graph of

$\nabla\varphi$. By the Knott-Smith criterion, it will be a solution of the more general problem

$$\min\left\{\int_{X\times Y}|x-y|^2\,d\pi(x,y);\ \pi\in\Pi(\mu,\nu)\right\}, \tag{3.17}$$

which is exactly the Kantorovich minimization problem with quadratic cost between μ and ν.

2. Let us check that the optimal transportation theorem applies. By assumption, the measures μ, ν do not give mass to small sets, and

$$\int_X |x|^2\,d\mu(x)=\int_X |x|^2\,d(h\#\lambda)(x)=\int_W |h(w)|^2\,d\lambda(w)<+\infty.$$

So, by Theorem 2.12, there exists a pair (φ,φ^*) of convex conjugate maps, whose gradients are inverse to each other in the almost everywhere sense on X, Y, and uniquely determined almost everywhere, which solve the dual Monge-Kantorovich problem formulated in (2.9). Then,

$$\nu=\nabla\varphi\#\mu=\nabla\varphi\#(h\#\lambda)=(\nabla\varphi\circ h)\#\lambda.$$

So $s=\nabla\varphi\circ h$ belongs to $S(W,Y)$, and $\pi=(h\times s)\#\lambda$ is the unique solution of the minimization problem (3.17); thus $(\nabla\varphi,s)$ also solves the minimization problem (3.16).

3. Since $\nabla\varphi^*(\nabla\varphi(x))=x$ for $d\mu$-almost every x, and since $h\#\lambda=\mu$, we see that

$$\nabla\varphi^*\big(\nabla\varphi(h(w))\big)=h(w)\qquad\text{for } d\lambda\text{-almost all } w.$$

In particular, by setting $\psi=\varphi^*$, we have constructed a solution $(\nabla\psi,s)$ of the polar decomposition problem.

4. By the arguments above, the uniqueness in the polar factorization theorem is a consequence of the uniqueness of the L^2 projection s of h onto $S(\Omega)$, which we now proceed to prove. Let s' be another L^2 projection of h. Then, uniqueness in the Monge-Kantorovich problem implies that

$$(h\times s)\#\lambda=(h\times s')\#\lambda. \tag{3.18}$$

This does *not* imply that $s'=s$ (a counterexample was communicated to us by Ambrosio). However, here we have some additional information, due to the fact that

$$s=\nabla\varphi^*\circ h. \tag{3.19}$$

We shall exploit this as follows: let $F(x,y)=\nabla\varphi^*(x)\cdot y$; then, by the definition of image measure and (3.18),

$$\int_W F\big(h(w),s(w)\big)\,d\lambda(w)=\int_W F\big(h(w),s'(w)\big)\,d\lambda(w).$$

Taking (3.19) into account, this particularizes to

$$\int_W |s(w)|^2 \, d\lambda(w) = \int_W s(w) \cdot s'(w) \, d\lambda(w).$$

Since on the other hand s and s' have the same $L^2(d\lambda)$ norm, it results that $s = s'$, $d\lambda$-almost everywhere. □

Remark 3.17. The dual problem corresponding to the minimization problem (3.16) is

$$\sup \left\{ \int_Y \left[\varphi(u(y)) + \psi(y)\right] d\nu(y); \quad \varphi(x) + \psi(y) \leq c(x,y) = |x-y|^2 \right\}.$$

3.4. Related facts

The material here is mostly extracted from [**59**]. It will not be used later, and is only intended to provide a better intuition of the polar factorization theorem.

First of all, Brenier's theorem is in some sense a natural generalization of a well-known theorem of monotone rearrangement on the line:

Theorem 3.18 (Monotone rearrangement theorem). *Let $h : [0,1] \to \mathbb{R}$ be an L^p function, $p \geq 1$. Then there exists a unique nondecreasing rearrangement $h^\#$ of h. Moreover, there exists a measure-preserving map $s : [0,1] \to [0,1]$ such that $h = h^\# \circ s$.*

Remark 3.19. In this context, the nondegeneracy condition (3.3) means that no level set of h has positive measure. If it is not satisfied, then there is no uniqueness of s in Theorem 3.18: one can always rearrange h arbitrarily on a level set of positive measure. Conversely, if no level set of h has positive measure, then there is uniqueness of s.

But Brenier's theorem also unifies several known facts which were apparently loosely related, if at all. In particular,

(a) **The polar factorization of real matrices:** Any matrix $M \in M_n(\mathbb{R})$ can be written as $M = SO$, where S is symmetric nonnegative, and O is an orthogonal matrix ($S \in S_n^+(\mathbb{R}), O \in O_n(\mathbb{R})$).

(b) **The Helmholtz decomposition of vector fields:** Any L^2 vector field w in a (reasonably smooth) open set $\Omega \subset \mathbb{R}^n$ can be written uniquely as

$$w = v + \nabla p,$$

where v is divergence-free ($\nabla \cdot v = 0$), tangent to $\partial\Omega$, and p is a real-valued function (or distribution) on Ω.

Let us expand briefly on points (a) and (b).

3.4.1. Polar factorization of matrices. To see the connection between the polar factorization of mappings and the polar factorization of matrices, it suffices to note that $M_n(\mathbb{R})$ can be embedded isometrically (up to a multiplicative factor) into $L^2(B(0,1),\mathbb{R}^n)$, where $B(0,1)$ is the unit ball in $\mathbb{R}^n$, by

$$M \longmapsto [x \mapsto Mx].$$

Here $M_n(\mathbb{R})$ is endowed with the Hilbert-Schmidt norm $\|\cdot\|_{HS}$, defined by

$$\|M\|_{HS}^2 = \operatorname{tr}(M^T M) = \sum_{ij} m_{ij}^2, \qquad M = (m_{ij}).$$

Then one can check that $O_n(\mathbb{R}) \subset S(B(0,1))$, while symmetric matrices can be viewed as gradients of quadratic functions (exercise). In particular, we realize that the factor O, in the factorization $M = SO$, may be characterized as *the projection of M onto the group $O_n(\mathbb{R})$*. In fact, one can state the matrix polar factorization in the following way:

Proposition 3.20 (Matrix polar factorization). *Let $M \in M_n(\mathbb{R})$. Then there exist $O \in O_n(\mathbb{R})$ and $S \in S_n^+(\mathbb{R})$ such that $M = SO$. Moreover, the admissible matrices O in this decomposition are the orthogonal projections of M onto $O_n(\mathbb{R})$:*

$$(3.20)\quad MO^{-1} \in S_n^+(\mathbb{R}) \Longleftrightarrow \left[\forall\, \widetilde{O} \in O_n(\mathbb{R}), \quad \|M - O\|_{HS} \le \|M - \widetilde{O}\|_{HS}\right].$$

Moreover, if M is invertible, then this decomposition is unique.

Sketch of proof of (3.20). 1. Since $\|\widetilde{O}\|_{HS}^2 = n$, the inequality on the right-hand side is equivalent to $\operatorname{tr}(M^T O) \ge \operatorname{tr}(M^T \widetilde{O})$. But $\operatorname{tr}(M^T O) = \operatorname{tr}(O^T M) = \operatorname{tr}(MO^T) = \operatorname{tr}(MO^{-1})$, and similarly $\operatorname{tr}(M^T\widetilde{O}) = \operatorname{tr}(\widetilde{O}^T M) = \operatorname{tr}(O\widetilde{O}^T MO^{-1}) = \operatorname{tr}((MO^{-1})(O\widetilde{O}^T))$. Since $O\widetilde{O}^T$ is an arbitrary element of $O_n(\mathbb{R})$, it is actually equivalent to prove that, for any $S \in M_n(\mathbb{R})$,

$$(3.21)\qquad S \in S_n^+(\mathbb{R}) \Longleftrightarrow \left[\forall\, O \in O_n(\mathbb{R}), \quad \operatorname{tr} S \ge \operatorname{tr}(SO)\right].$$

2. For the direct implication in (3.21), diagonalize S and use the fact that all coefficients of O are bounded by 1 in absolute value.

3. For the converse implication, choose $O = I_n + \varepsilon A + O(\varepsilon^2)$, where A is an arbitrary antisymmetric matrix (recall, or show as an exercise, that the tangent plane to $O_n(\mathbb{R})$ at I_n can be identified with the space $A_n(\mathbb{R})$ of antisymmetric matrices). By letting $\varepsilon \to 0$, we deduce that $\operatorname{tr}(SA) = 0$; and this holds for all $A \in A_n(\mathbb{R})$. Using the orthogonal decomposition $M_n(\mathbb{R}) = S_n(\mathbb{R}) + A_n(\mathbb{R})$, one proves that $S \in S_n(\mathbb{R})$. By diagonalizing S, one sees that all its eigenvalues are nonnegative, which concludes the argument. ☐

3.4.2. The Helmholtz decomposition theorem. Brenier's theorem can also be seen as a nonlinear version of the Helmholtz decomposition. Or, to state things more rigorously, the Helmholtz decomposition can be seen as a linearized version of Brenier's theorem (with the improvement that the nondegeneracy condition is no longer necessary). Let us explain this in an informal way.

Let v be a vector field on Ω; then one can formally consider a path $z(\cdot)$ in $L^2(\Omega;\mathbb{R}^n)$ of the form $z(\varepsilon) = \mathrm{Id} + \varepsilon v + o(\varepsilon)$. If ε is small and everything is smooth, then $z(\varepsilon)$ satisfies the nondegeneracy condition, so one can factor it and write $z(\varepsilon) = \nabla\psi(\varepsilon)\circ s(\varepsilon)$. It is natural to look for $\psi(\varepsilon)$ as $|x|^2/2+\varepsilon p+o(\varepsilon)$, and for $s(\varepsilon)$ as $\mathrm{Id}+\varepsilon w+o(\varepsilon)$, where w is "tangent to the space of measure-preserving maps at the identity". But this last condition precisely means that the vector field w is tangent to the boundary $\partial\Omega$ and divergence-free ($\nabla\cdot w = 0$). Then, by formally identifying the expansions of $z(\varepsilon)$ and of $\nabla\psi(\varepsilon)\circ s(\varepsilon)$, one arrives at

$$v = w + \nabla p.$$

Remark 3.21. Note that the same heuristics, applied with the polar factorization of matrices, yield the orthogonal decomposition $M_n(\mathbb{R}) = S_n(\mathbb{R}) + A_n(\mathbb{R})$.

Chapter 4

The Monge-Ampère Equation

In Chapter 2 we were interested in the existence of an optimal transportation; now we shall become interested in its *regularity*. Under what conditions on the probability measures will the optimal transportation be smooth? So far this problem has been solved only for a quadratic cost function in Euclidean space. Indeed, as we shall see, the convex function φ provided by Theorem 2.12 turns out to be a solution, at least in a generalized sense, of an equation of Monge-Ampère type, for which a regularity theory has been constructed by Caffarelli and by Urbas. The techniques involved in this study are really intricate and, to a large extent, disconnected from the rest of these notes; this is why we shall touch the subject from a very superficial point of view. The reader who would like to know more about this is advised to consult the research papers mentioned below, or the set of notes [**255**] by Urbas.

Independently of regularity issues, it is also of interest to know in which sense φ solves the Monge-Ampère equation. We shall discuss below some useful results on that issue, due to Caffarelli and McCann, which apply even in nonsmooth contexts.

4.1. Informal presentation

4.1.1. The Monge-Ampère equation. Let $d\mu(x) = f(x)\,dx$, $d\nu(y) = g(y)\,dy$ be two probability measures, absolutely continuous with respect to Lebesgue measure. As we know from Theorem 2.12, there exists a ($d\mu$-almost everywhere) unique gradient of a convex function, $\nabla\varphi$, such that for

all test functions $\zeta \in C_b(\mathbb{R}^N)$,

$$\int \zeta(y) g(y)\, dy = \int \zeta(\nabla\varphi(x))\, f(x)\, dx. \tag{4.1}$$

Let us assume that $\nabla\varphi$ is smooth (say, C^1) and one-to-one (which is the case if φ is strictly convex). Then we can perform the change of variables $y = \nabla\varphi(x)$ in the left-hand side of (4.1):

$$\int \zeta(y) g(y)\, dy = \int \zeta(\nabla\varphi(x))\, g(\nabla\varphi(x))\, \det(D^2\varphi(x))\, dx. \tag{4.2}$$

Since ζ is arbitrary, the combination of (4.1) and (4.2) yields

$$f(x) = g(\nabla\varphi(x)) \det D^2\varphi(x). \tag{4.3}$$

If g is positive, this can also be rewritten as

$$\det D^2\varphi(x) = \frac{f(x)}{g(\nabla\varphi(x))}. \tag{4.4}$$

This is a particular case of the general **Monge-Ampère equation**

$$\det D^2\varphi(x) = F\big(x, \varphi(x), \nabla\varphi(x)\big). \tag{4.5}$$

The study of solutions to equations of the form (4.5) is an old topic; a lot of information about them can be found in the book by Gutiérrez [**164**]. A famous particular case of equation (4.4) is the so-called equation of **prescribed Gauss curvature**,

$$\det D^2\varphi(x) = \kappa(x)\big(1 + |\nabla\varphi(x)|^2\big)^{\frac{n+2}{2}}, \qquad x \in \mathbb{R}^n,$$

where κ is a given function. Indeed, if φ solves this equation, then the graph of φ (viewed as a subset of $\mathbb{R}^{n+1}$) has scalar curvature $\kappa(x)$ at each point $(x, \varphi(x))$. Equations of this type have been studied since Aleksandrov [**7, 8**] in the forties. At the present time, a lot of energy is being devoted to the study of generalizations called equations of prescribed k-th curvature, see Urbas [**259, 260**] and references therein.

4.1.2. Linearization. Monge-Ampère equations can be connected with Laplace-type equations, more precisely **linear second-order elliptic equations** of the form

$$\sum_{ij} a_{ij} \frac{\partial^2\varphi}{\partial x_i\, \partial x_j} + \sum_i a_i \frac{\partial\varphi}{\partial x_i} + c\varphi = h,$$

where $(a_{ij}(x))_{1\le i,j\le n}$ is a positive definite, $n \times n$ matrix-valued function, $(a_i(x))_{1\le i\le n}$ a vector-valued function, and c and h are scalar functions. The Laplace equation is the particular case $\Delta\varphi = h$. Linear second-order elliptic equations are among the partial differential equations which have been most

intensively studied, see [**155**] and the numerous references therein. A pedagogical introduction to their theory can be found in [**133**]. To see the link with equation (4.5), think that $\det D^2\varphi$ is the product of the eigenvalues of the Hessian $D^2\varphi$, while $\Delta\varphi$ is the sum of these eigenvalues. Therefore, Laplace equations can be seen as linearized versions of Monge-Ampère equations. The following exercise makes this a little bit more concrete.

Exercise 4.1 (Linearization of the Monge-Ampère equation). Assume that f is strictly positive, and that φ is very close to the identity; accordingly, g is very close to f. Make the ansatz

$$\varphi(x) = \varphi_\varepsilon(x) = \frac{|x|^2}{2} + \varepsilon\psi + O(\varepsilon^2), \qquad g = g_\varepsilon = \big(1 + \varepsilon h + O(\varepsilon^2)\big) f,$$

plug this into formula (4.3) and keep only first-order terms in ε. Check that this linearization procedure formally turns the Monge-Ampère equation into the linear equation

$$L\psi = h, \tag{4.6}$$

where

$$L = -\Delta + \nabla(-\log f) \cdot \nabla.$$

It follows from the standard theory of elliptic equations that ψ in (4.6) is smooth as soon as h and f are, provided that f is strictly positive. But the nonlinear problem (4.3) turns out to be considerably more tricky.

4.1.3. Further considerations. The Monge-Ampère equation belongs to the class of so-called **fully nonlinear elliptic equations**. Here is one among several possible general definitions:

Definition 4.2 (Ellipticity of a nonlinear equation). Let G be a continuous function on a subset D of $\mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \times S_n(\mathbb{R})$, convex with respect to the last (matrix) variable. The equation

$$G(x, \varphi, \nabla\varphi, D^2\varphi) = 0 \tag{4.7}$$

is said to be elliptic if, for all choices of x, r, p, X, Y,

$$Y \geq X \Longrightarrow G(x, r, p, Y) - G(x, r, p, X) \geq 0.$$

It is said to be uniformly elliptic if there exist positive numbers $\lambda, \Lambda > 0$ such that, for all choices of x, r, p, X, Y,

$$Y \geq X \Longrightarrow \Lambda \operatorname{tr}(Y - X) \geq G(x, r, p, Y) - G(x, r, p, X) \geq \lambda \operatorname{tr}(Y - X).$$

Most of the time, one studies equation (4.7) for x belonging to a subdomain Ω of $\mathbb{R}^n$, for instance a smooth bounded open set. Then equation (4.7) has to be supplemented with some boundary conditions, the simplest of which is the **Dirichlet boundary condition**, namely $\varphi \equiv 0$ on $\partial\Omega$.

The theory of uniformly elliptic equations is by now quite well-developed, see Caffarelli and Cabré [**80**]. In full generality, the Monge-Ampère equation is not elliptic; but it is if one restricts it to the set of convex φ's. On the other hand, it is not uniformly elliptic. One of the main difficulties in the subject is precisely to control the degeneracy of the ellipticity, and rule out the possibility that φ solves the Monge-Ampère equation with some singular eigenvalue being compensated by some zero eigenvalues.

Among natural function spaces for the study of elliptic equations, or partial differential equations in general, are the spaces $W^{k,p}(\Omega)$ and $C^{k,\alpha}(\Omega)$. The former is made of functions on Ω having all their partial derivatives up to order k in $L^p(\Omega)$; and the latter is made of functions on Ω having all their derivatives up to order k in the space $C^{0,\alpha}(\Omega)$ of α-Hölder continuous functions. By analogy with the theory of the Laplace equation, one may a priori think that $\det D^2u \in W^{k,p}$ implies $u \in W^{k+2,p}$; or that $\det D^2u \in C^{k,\alpha}$ implies $u \in C^{k+2,\alpha}$. In the end this turns out to be true under certain restrictions, but this theory is much more intricate than the theory of the Laplace equation. Let us describe some of the main difficulties.

First, the Monge-Ampère equation has a very rich family of invariants. Consider the simplest Monge-Ampère equation,

$$\det D^2\varphi = 1. \tag{4.8}$$

Then the set of solutions is invariant under the action of rotations, well-chosen dilations, or any affine transformation with unit determinant. For instance, if φ solves (4.8) on $\mathbb{R}^2$, then so does $\varphi(\varepsilon x, y/\varepsilon)$, for all $\varepsilon > 0$.

This simple remark shows (exercise) that for equation (4.8) there cannot exist any interior a priori estimates of the kind which exist for Laplace equations, and which are the first step towards the complete regularity theory. By an **interior** a priori estimate, we mean a regularity estimate which would hold on any bounded open domain Ω' such that $\overline{\Omega'}$ is included within a larger open domain Ω, on which φ solves the equation. Here is a typical interior estimate [**133**, p. 29]: solutions of the Laplace equation $\Delta u = 0$ in $\Omega \subset \mathbb{R}^n$ satisfy $\|u\|_{C^k(\Omega')} \le C_k \|u\|_{L^1(\Omega)}$ for all k, where C_k depends only on k, n and the distance between $\partial\Omega$ and $\partial\Omega'$.

Thus we see that for solutions of Monge-Ampère equations, the boundary behavior (i.e. the behavior of φ at $\partial\Omega$) has to enter the regularity estimates.

Another obstruction to smoothness estimates is the possibility of non-uniform convexity, in dimension $n \ge 3$. For example, Pogorelov noticed that the non-uniformly convex function

$$\varphi(x) = (1 + x_n^2)\left(\sum_{k=1}^{n-1} x_k^2\right)^{1-\frac{1}{n}}$$

satisfies $\det D^2\varphi \in C^\infty$ when $n \geq 3$, but only lies in $C^{1,1-\frac{2}{n}}$. As a variant, Caffarelli produced an even more frightening example: the function

$$\varphi(x) = \left(\sum_{k=1}^{n-1} x_k^2\right)^{\frac{1}{2}} + (1 + x_n^2)\left(\sum_{k=1}^{n-1} x_k^2\right)^{\frac{4}{n}}$$

is such that $\det D^2\varphi$ is positive Lipschitz if $n = 3$, positive analytic if $n = 4$, but φ is not better than Lipschitz!

It turns out that these two obstructions are the only ones. For instance, there exist some regularity results that take into account the behavior of the solution at the boundary of the domain, and the strict convexity of the domain as well. Let us say that Ω is a **uniformly convex** open set of $\mathbb{R}^n$ if it is defined by an equation of the form $\psi < 0$ for some ψ with $D^2\psi \geq \lambda I_n$ ($\lambda > 0$). Such sets are automatically bounded. The following result is known: if Ω is a uniformly convex open subset of $\mathbb{R}^n$ and F is a C^∞ positive function, then the equation $\det D^2\varphi = F(x, \varphi, \nabla\varphi)$, with Dirichlet boundary condition, has a unique C^∞ solution.

4.1.4. Various notions of weak solutions. To arrive at (4.4), we assumed that $\nabla\varphi$ was C^1, i.e. $\varphi \in C^2$. But we still do not know whether this is true, and we would like to study equation (4.4) without this a priori assumption. Since φ is convex, it is continuous and locally Lipschitz on the interior of its domain, but not necessarily twice differentiable. So a problem immediately arises: how to define $\det D^2\varphi$ if φ is not a priori smooth? And consequently, how to make sense of equation (4.5)? There are basically three possibilities.

(i) Aleksandrov solutions: This concept is defined in terms of the **Hessian measure** associated to φ, which we denote by $\det_{\mathrm{H}} D^2\varphi$. This is a Borel measure defined in the following way: for any measurable set $E \subset \mathbb{R}^n$,

$$\det\nolimits_{\mathrm{H}} D^2\varphi[E] = |\partial\varphi(E)|,$$

where

$$\partial\varphi(E) = \bigcup_{x\in E} \partial\varphi(x).$$

By definition, φ is an Aleksandrov solution of (4.5) if the Hessian measure $\det_{\mathrm{H}} D^2\varphi$ is absolutely continuous with respect to Lebesgue measure, and its density coincides with the right-hand side of (4.5), which is defined almost everywhere.

An equivalent formulation is that the measure $\det_{\mathrm{H}} D^2\varphi$ has no singular part, and that the Monge-Ampère equation (4.5) holds almost everywhere, with $\det D^2\varphi$ defined as $\det D_A^2\varphi$ (the determinant of the Aleksandrov second derivative, which exists almost everywhere).

(ii) Viscosity solutions: This concept is in fact equivalent to the preceding, but may be simpler to use in certain situations. It is the usual concept of viscosity solutions (see for instance [**133**]), up to minor modifications. By definition, φ is a viscosity solution of (4.5), set in an open set Ω, if

1) whenever ψ is a convex C^2 test function such that $\varphi - \psi$ has a strict local maximum at x_0, then $\det D^2\psi(x_0) \geq F(x_0, \varphi(x_0), \nabla\psi(x_0))$;

2) whenever ψ is a convex C^2 test function such that $\varphi - \psi$ has a strict local minimum at x_0, then $\det D^2\psi(x_0) \leq F(x_0, \varphi(x_0), \nabla\psi(x_0))$.

Of course there is no ambiguity in the significance of $\det D^2\psi(x_0)$, since ψ is smooth.

(iii) Brenier solutions: This concept is strictly weaker and only applies to the equation which is of interest for us, namely (4.4) — not to equation (4.5). By definition, φ is a Brenier solution of equation (4.4) if $\nabla\varphi\#\mu = \nu$, where μ and ν are the probability measures defined by $d\mu(x) = f(x)\,dx$, $d\nu(y) = g(y)\,dy$.

Instructive Exercise 4.3 (Smooth weak solutions are strong). When φ is twice continuously differentiable, show that all three concepts reduce to the Monge-Ampère equation $\det D^2\varphi(x) = F(x, \varphi(x), \nabla\varphi(x))$ in the usual sense.

Remarks 4.4. (i) A clever use of the different notions of solutions can be quite powerful, as exemplified by Problem 7 in Chapter 10.

(ii) One of the original features about the Monge-Ampère equation which arises from optimal transportation is that there is no boundary condition, except that $\nabla\varphi$ should map the support of f onto the support of g.

It may happen that a Brenier solution is not a solution in the Aleksandrov sense, because $\det_{\mathrm{H}} D^2\varphi$ has a singular part. An explicit counterexample is quite easy to construct (see Figure 4.1): consider the case when f is the characteristic function of a ball, and g is the characteristic function of the union of two half-balls.

Exercise 4.5. Compute the Hessian measure $\det_{\mathrm{H}} D^2\varphi$ in this example, and show that it carries a singular part along the line of separation.

One may however think that this kind of counterexamples can be ruled out by an assumption of connectedness for the support of g. But this is not true: by a perturbative argument [**75**], it is easy to construct counterexamples in which the support of g is smooth and connected. For this it suffices to build a very thin bridge between the two half-balls, with thickness ε,

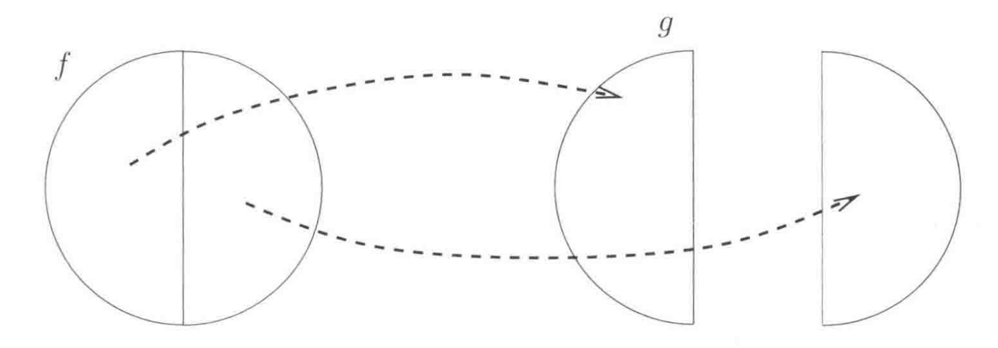


Figure 4.1. A singular Brenier solution: $\varphi(x_1, x_2) = (x_1^2 + x_2^2)/2 + |x_1|$

then regularize the resulting support a little bit by modifying it on a thin shell of size ε. Let φ_ε be the corresponding convex function. By the general inequality

$$\det_{\mathrm{H}} D^2\varphi \leq \liminf_{\varepsilon\to 0} \det_{\mathrm{H}} D^2\varphi_\varepsilon,$$

where the lim inf is to be understood in the weak-measure sense, one can show (exercise) that $\det_{\mathrm{H}} D^2\varphi_\varepsilon$ should develop a singular part as $\varepsilon \to 0$.

A better interpretation for this counterexample is the *nonconvexity* of the support of g. The point is that the image of the support of f by the map $\partial\varphi$ has to fill (at least) the convex envelope of the support of g, and that we lose the control of the Hessian measure at all the points x such that $\nabla\varphi(x)$ does not lie inside the support of g.

This remark makes it natural to develop the regularity theory under the assumption that the support of g, the *"target"*, is convex.

4.2. Regularity

In this section, we investigate in more detail the Monge-Ampère equation (4.3) in the context of mass transportation. Throughout this section, we shall only consider the Monge-Kantorovich mass transportation problem between probability measures μ and ν which are *absolutely continuous* with respect to Lebesgue measure, with respective densities f and g.

4.2.1. Nonsmooth setting. Let us first consider equation (4.3) without imposing smoothness conditions on the densities. This situation has been studied by Caffarelli [**75**] and by McCann [**203**]. To avoid confusion, it will be useful to denote the Aleksandrov second derivative by $D_A^2\varphi$. This will allow us to clearly distinguish between the nonnegative function $\det D_A^2\varphi$ and

the Hessian measure $\det_{\mathrm{H}} D^2\varphi$. As we shall see, under suitable assumptions, the former is the density of the absolutely continuous part of the latter.

We begin with a useful lemma.

Lemma 4.6 (Push-forward in terms of subdifferential). *(i) Let φ be a convex function and μ a probability measure on $\mathbb{R}^n$, absolutely continuous with respect to Lebesgue measure. Then for all Borel sets $A \subset \mathbb{R}^n$,*

$$\nabla\varphi\#\mu\,[A] = \mu[\partial\varphi^*(A)].$$

(ii) Further assume that $\nabla\varphi\#\mu = \nu$ is absolutely continuous with respect to Lebesgue measure, and denote by f and g the respective Lebesgue densities of μ and ν. Then, for all Borel sets $A \subset \mathbb{R}^n$,

$$\int_{\partial\varphi(A)} g(y)\,dy = \int_A f(x)\,dx.$$

Proof. 1. To prove (i), it is sufficient to prove that for any measurable set A, one has $\mu[(\nabla\varphi)^{-1}(A)] = \mu[\partial\varphi^*(A)]$. From general properties of convex functions (recall subsection 2.1.3 in Chapter 2), $\nabla\varphi(x) = y \Longrightarrow x \in \partial\varphi^*(y)$; consequently, $(\nabla\varphi)^{-1}(A) \subset \partial\varphi^*(A)$. Since μ is absolutely continuous, it suffices to check that the set

$$Z = \partial\varphi^*(A) \setminus (\nabla\varphi)^{-1}(A)$$

has zero Lebesgue measure.

2. Let $z \in \partial\varphi^*(A)$; then there exists $x \in A$ such that $z \in \partial\varphi^*(x)$, which means $x \in \partial\varphi(z)$. If z is a differentiability point of φ, then necessarily $\nabla\varphi(z) = x \in A$, so $z \in (\nabla\varphi)^{-1}(A)$. Hence Z is included in the set of points where φ is not differentiable, so it has zero Lebesgue measure. This proves part (i) of the lemma.

3. Recall from Theorem 2.12 (iv) that $\nabla\varphi^*\#\nu = \mu$ if ν is absolutely continuous. Taking this into account, point (ii) of the lemma is a direct consequence of point (i). □

To go further, we shall use the concept of **Lebesgue density point**, defined in the following proposition, whose proof can be found, together with more details, in Rudin [**231**, pp. 138–143]:

Proposition 4.7 (Lebesgue's density theorem). *Let $d\mu(x) = f(x)\,dx + d\sigma(x)$ be a Borel measure on an open subset Ω of $\mathbb{R}^n$, where $f(x)\,dx$ is the absolutely continuous part of μ, and σ is concentrated on a set of Lebesgue measure 0. We assume that $\mu[K] < +\infty$ for all compact sets $K \subset \Omega$. Then,*

(i) For almost all $x \in \mathbb{R}^n$,

$$\frac{\mu[B_r(x)]}{|B_r(x)|} \xrightarrow[r\to 0]{} f(x). \tag{4.9}$$

Those points x satisfying (4.9) are called Lebesgue (density) points of μ, or Lebesgue points of f.

(ii) The same conclusion holds true if the balls B_r are replaced by a family of sets $(C_k)_{k\in\mathbb{N}}$ such that $\bigcap C_k = \{x\}$, and there exist two families of balls (B_k) and (B'_k) such that $B_k \subset C_k \subset B'_k$, and $|B_k|/|B'_k|$ is bounded from below by a positive constant, uniformly in k.

Lebesgue's density theorem is at the basis of the following very useful result, due to McCann [**203**].

Theorem 4.8 (The Monge-Ampère equation holds true almost everywhere). *Let μ and ν be two probability measures on $\mathbb{R}^n$, absolutely continuous with respect to Lebesgue measure, and let φ be a convex function such that $\nabla\varphi\#\mu = \nu$. Let f and g be the respective densities of μ and ν, and let $\Omega = \mathrm{Int}(\mathrm{Dom}(\varphi))$. Let $\det D^2_A\varphi$ be the determinant of the Hessian of φ, in the Aleksandrov sense; this is a nonnegative function in $L^1_{\mathrm{loc}}(\Omega)$, well-defined almost everywhere. Let $M \subset \Omega$ be the set of points in Ω where $D^2_A\varphi$ is defined and invertible, which are also Lebesgue points for $\det D^2_A\varphi$. Then,*

(i) M is of full measure for μ, and $\partial\varphi(M)$ is of full measure for ν.

(ii) The measure $\det D^2_A\varphi(x)\,dx$ coincides with the absolutely continuous part of the Hessian measure $\det_{\mathrm{H}} D^2\varphi$, is concentrated on M and satisfies the push-forward formula

$$\nabla\varphi\#\big[\det D^2_A\varphi(x)\,dx\big] = 1_{\partial\varphi(M)}\,dx.$$

(iii) For almost all $x \in \mathbb{R}^n$, we have the Monge-Ampère equation

$$\det D^2_A\varphi(x)\,g(\nabla\varphi(x)) = f(x). \tag{4.10}$$

(iv) Whenever U is a nonnegative measurable function on $\mathbb{R}_+$ such that $U(0) = 0$,

$$\int_{\mathbb{R}^n} U(g(y))\,dy = \int_{\mathbb{R}^n} U\left(\frac{f(x)}{\det D^2_A\varphi(x)}\right)\det D^2_A\varphi(x)\,dx. \tag{4.11}$$

Remarks 4.9. (i) Of course, the existence of $D^2_A\varphi(x)$ implies the existence of $\nabla\varphi(x)$; so φ is differentiable on the whole of M.

(ii) It is a good exercise to check property (ii) directly when φ is well-behaved.

(iii) Conclusion (iv) can immediately be extended to any measurable function U satisfying $U(0) = 0$, and either $U \geq V$ or $U \leq V$, where $V(0) = 0$, $V \circ g \in L^1(\mathbb{R}^n)$. Indeed, in that case one can apply point (iv) to $\pm(U - V)$, V_+ and V_- separately. This allows one to extend the validity of (4.11) to basically all situations in which $\int U(g(y))\, dy$ is well-defined in $\mathbb{R} \cup \{\pm\infty\}$. It was pointed out to us by Ambrosio that this part of the theorem could also be deduced from the theory of change of variables by functions of bounded variation.

Proof of Theorem 4.8. The proof of this theorem is rather elementary but a little bit tricky and tedious. It may be omitted at first reading.

1. First, it is a general consequence of the convexity of φ that for any compact set $K \subset \Omega$,

$$\det\nolimits_{\mathrm{H}} D^2\varphi[K] = |\partial\varphi(K)| < +\infty.$$

So $\det_{\mathrm{H}} D^2\varphi$ is a locally finite measure, and Lebesgue's density theorem applies.

2. By Aleksandrov's theorem, we know that $\det D^2_A\varphi(x)$ is well-defined almost everywhere in Ω. We now check that $d\lambda(x) = \det D^2_A\varphi(x)\, dx$ is the absolutely continuous part of the Hessian measure $\det_{\mathrm{H}} D^2\varphi$. In view of Lebesgue's density theorem, it is sufficient to prove that for almost all $x \in \Omega$,

$$\frac{\det_{\mathrm{H}} D^2\varphi[B_r(x)]}{|B_r(x)|} \xrightarrow[r\to 0]{} \det D^2_A\varphi(x);$$

in other words,

$$\frac{|\partial\varphi(B_r(x))|}{|B_r(x)|} \xrightarrow[r\to 0]{} \det D^2_A\varphi(x).$$

But this is precisely equation (2.23). This proves the first part of (ii).

3. Since $\det_{\mathrm{H}} D^2\varphi$ is locally finite, and $\det D^2_A\varphi$ is the density of its absolutely continuous part, we know that $\det D^2_A\varphi$ is locally integrable. We also know that almost all points in Ω are density points for $\det D^2_A\varphi$. To prove that M has full measure for μ, recall from subsection 2.1.3 in Chapter 2 that the set of points x where $D^2_A\varphi(x)$ is not invertible is included in $\partial\varphi^*(C)$, where C is the set of points y in $\mathrm{Dom}(\varphi^*)$ such that $D^2_A\varphi^*(y)$ is not defined. By Aleksandrov's theorem, C has zero Lebesgue measure. Since $\nabla\varphi^*\#\nu = \mu$, and μ, ν are absolutely continuous, we can apply Lemma 4.6 to conclude that $\mu[\partial\varphi^*(C)] = \nu[C] = 0$. So M is of full measure for μ. By Lemma 4.6 again, $\nu[\partial\varphi(M)] = \mu[M] = 1$. This concludes the proof of (i).

4. Whenever $D^2_A\varphi(x)$ is not invertible, we have $\det(D^2_A\varphi(x)) = 0$, so λ gives zero measure to the set of such points x. Therefore λ is concentrated on M; this proves the second part of (ii).

5. We now check that $\nabla\varphi\#\lambda$ is absolutely continuous. Let $A \subset \partial\varphi(M)$, with $|A| = 0$. Since λ is absolutely continuous, it is concentrated on the set of points where φ is differentiable. By using the definition of Hessian measure and the fact that $\partial\varphi(x) = \{\nabla\varphi(x)\}$ whenever x is a differentiability point of φ (in particular when $x \in M$), we can write

$$\nabla\varphi\#\lambda[A] = \lambda[(\nabla\varphi)^{-1}(A)] \leq \det{}_{\mathrm{H}} D^2\varphi[(\nabla\varphi)^{-1}(A)] = |\partial\varphi(\nabla\varphi)^{-1}(A)|$$
$$= |\nabla\varphi(\nabla\varphi)^{-1}(A)| = |A| = 0.$$

So $\nabla\varphi\#\lambda$ is indeed absolutely continuous.

6. To conclude the proof of (ii), it only remains to check that the density of $\nabla\varphi\#\lambda$, at almost each $y \in \partial\varphi(M)$, is 1. Let $y \in \partial\varphi(M)$. We know that there exists $x \in M$ such that $\nabla\varphi(x) = y$; and since x lies in M, we know that $D^2_A\varphi(x)$ is well-defined and invertible. By the reminders in subsection 2.1.3 in Chapter 2,

$$\frac{|\partial\varphi^*(B_r(y))|}{|B_r(y)|} \xrightarrow[r\to 0]{} [\det D^2_A\varphi(x)]^{-1}. \tag{4.12}$$

Moreover, we can find a sequence $(r_k)_{k\in\mathbb{N}}$, converging to 0, such that B_{r_k} is not too much distorted by the action of $\partial\varphi^*$, in the sense that there exist balls B_k and B'_k with $B_k \subset \partial\varphi^*(B_{r_k}) \subset B'_k$, and $|B_k|/|B'_k|$ is bounded from below. Thus we can apply Lebesgue's density theorem to the L^1_{loc} function $\det D^2_A\varphi$ with the family of sets $\partial\varphi^*(B_{r_k})$; since x is a Lebesgue point of $\det D^2_A\varphi$,

$$\frac{1}{|\partial\varphi^*(B_{r_k}(y))|}\int_{\partial\varphi^*(B_{r_k}(y))} \det D^2_A\varphi \xrightarrow[k\to\infty]{} \det D^2_A\varphi(x). \tag{4.13}$$

If we multiply (4.12) with (4.13), and use the identity $\nabla\varphi\#\lambda[A] = \lambda[\partial\varphi^*(A)]$ from Lemma 4.6 (i), we find that

$$\frac{\nabla\varphi\#\lambda[B_{r_k}(y)]}{|B_{r_k}(y)|} \xrightarrow[r_k\to 0]{} 1.$$

This holds true for almost all $y \in \partial\varphi(M)$, so the density of the measure $\nabla\varphi\#\lambda$ is identically equal to 1 on $\partial\varphi(M)$. This concludes the proof of (ii).

7. Let A be a Borel set in $\mathbb{R}^n$. As a consequence of (ii),

$$\int_{\partial\varphi(A)} g(y)\,dy = \int_{\partial\varphi(M)} 1_{y\in\partial\varphi(A)}\, g(y)\,dy$$
$$= \int_M 1_{\nabla\varphi(x)\in\partial\varphi(A)} g(\nabla\varphi(x)) \det D^2_A\varphi(x)\,dx.$$

Whenever $x \in M$, the second derivative $D^2_A\varphi(x)$ is invertible, and therefore φ^* is (twice) differentiable at $\nabla\varphi(x)$. It follows that $\partial\varphi^*(\nabla\varphi(x))$ is reduced

to $\{x\}$, and that no $x' \neq x$ can satisfy $\nabla\varphi(x) = \nabla\varphi(x')$. In particular, $\nabla\varphi(x) \in \partial\varphi(A)$ is equivalent to $x \in A$. Thus we conclude that

$$\int_{\partial\varphi(A)} g(y)\,dy = \int_{A\cap M} g(\nabla\varphi(x)) \det D_A^2\varphi(x)\,dx. \tag{4.14}$$

Since λ is concentrated on M, in fact

$$\int_{\partial\varphi(A)} g(y)\,dy = \int_{A} g(\nabla\varphi(x)) \det D_A^2\varphi(x)\,dx. \tag{4.15}$$

Combining this with Lemma 4.6(i), we find that

$$\int_A f(x)\,dx = \int_{A} g(\nabla\varphi(x)) \det D_A^2\varphi(x)\,dx.$$

Since A is arbitrary, this implies point (iii).

8. From (ii) we know that

$$\int_{\partial\varphi(M)} U(g(y))\,dy = \int_{M} U(g(\nabla\varphi(x))) \det D_A^2\varphi(x)\,dx.$$

But for almost all $x \in M$, we can write, in view of (iii) and of the invertibility of $D_A^2\varphi$,

$$g(\nabla\varphi(x)) = \frac{f(x)}{\det D_A^2\varphi(x)}.$$

In particular,

$$\int_{\partial\varphi(M)} U(g(y))\,dy = \int_{M} U\left(\frac{f(x)}{\det D_A^2\varphi(x)}\right) \det D_A^2\varphi(x)\,dx.$$

Since M is of full measure for λ, the integral on the right-hand side can be extended to the whole of $\mathbb{R}^n$. And since $\partial\varphi(M)$ is of full measure for ν, we know that $g(y) = 0$ for almost all $y \in (\partial\varphi(M))^c$, in which case $U(g(y)) = 0$. Thus the integral on the left-hand side can also be extended to the whole of $\mathbb{R}^n$. This concludes the proof of (iv). □

Note carefully that the preceding theorem does not say anything about the behavior of the *singular part* of the Hessian measure. In particular, it does not imply that φ is a solution of the Monge-Ampère equation in the Aleksandrov sense. As we saw, there are counterexamples when the support of ν is not convex. The next theorem, due to Caffarelli, expresses the fact that, precisely, convexity is enough to rule out these problems and ensure that the Monge-Ampère equation is satisfied in the Aleksandrov sense.

Theorem 4.10 (Monge-Ampère equation in Aleksandrov sense for convex target). *Let μ and ν be two probability measures on $\mathbb{R}^n$, absolutely continuous with respect to Lebesgue measure. Let f and g be their respective densities, and let X and Y be their respective supports. Let φ be a convex*

function such that $\nabla\varphi\#\mu = \nu$. *Assume that* Y *is convex and that* g *is positive almost everywhere on* Y. *Then the Hessian measure* $\det_{\mathrm{H}} D^2\varphi$ *has no singular part on* X. *In particular,* φ *solves the Monge-Ampère equation* (4.3) *in the Aleksandrov sense.*

Proof. Going back to the definition of the Hessian measure, we see that we only have to prove

$$|N| = 0 \Longrightarrow |\partial\varphi(N)| = 0$$

for any measurable set $N \subset X$. So let N be a negligible subset of X; by part (ii) of Lemma 4.6,

$$\int_{\partial\varphi(N)} g(y)\,dy = \int_N f(x)\,dx = 0. \tag{4.16}$$

On the other hand, we know from Theorem 2.12 that $\nabla\varphi(X) \subset Y$, which implies

$$\partial\varphi(N) \subset \overline{\mathrm{Conv}\,(Y)},$$

where the right-hand side is the smallest closed convex set containing Y. By assumption this is just Y (this is the point where we use the convexity of the support of g!), and since g is positive almost everywhere on Y, we conclude from (4.16) that $|\partial\varphi(N)| = 0$. Thus the Hessian measure is really absolutely continuous on X. Since we already know from Theorem 4.8 that the Monge-Ampère equation is satisfied almost everywhere, we conclude that it is also satisfied in the Aleksandrov sense. □

Remark 4.11. The absolute continuity of the Hessian measure $\det_{\mathrm{H}} D^2\varphi$ does not a priori imply the absolute continuity of the *distributional* Hessian $D^2_{\mathcal{D}'}\varphi$ itself, as can be shown by the example of $\varphi(x_1, x_2) = |x_1|$. However, one may imagine that the assumption of ν being absolutely continuous with convex support is enough to rule out this type of counterexamples. To our knowledge, no result is known in this direction.

4.2.2. Caffarelli's regularity theory. In this subsection, we shall see how the general theory of Monge-Ampère equations gives some insight on the *smoothness* of the optimal transportation map when the densities themselves are smooth.

The standard references for this topic are the series of papers by Caffarelli [**71, 72, 73, 74, 75, 76, 78**], and the works by Urbas [**255, 256, 257, 258**]. The regularity of solutions to Monge-Ampère equations is in itself a rather old and important subject, pioneered by Aleksandrov [**9**] and Pogorelov [**222, 223**]. Much is known in dimension 2 [**236**], but higher dimensions are much more tricky. The survey paper [**255**] by Urbas is a useful reference for nonspecialist readers who would like to know more about this topic; see also the book by Gutiérrez [**164**]. The techniques of Caffarelli and

of Urbas are quite different. In the sequel we shall only explain Caffarelli's approach; one reason for this choice is the existence of the above-mentioned user-friendly survey by Urbas. No proofs will be provided here, but just a few hints of the main ideas.

As above, the final goal is the study of the regularity of solutions to

$$\det D^2\varphi(x) = \frac{f(x)}{g(\nabla\varphi(x))}. \tag{4.17}$$

The first step is the study of the inequality

$$0 < \lambda \le \det D^2_A\varphi \le \Lambda, \tag{4.18}$$

where λ, Λ are positive constants. Let us cut the graph of φ by a hyperplane H_0, and study the portion of the graph which lies below H_0. Adding a linear transformation to φ if necessary, we can assume that H_0 is of the form $(y = 0)$, and we wish to study the regularity of φ inside the convex set $(\varphi \le 0)$.

As we said above, it is impossible to hope for uniform a priori estimates on φ, due to the invariance group of the Monge-Ampère equation: maybe the set $\varphi^{-1}(0)$ is extremely stretched in a particular direction... However, since it is convex, the famous **John's lemma** (a proof of which can be found in [**255**]) asserts the existence of a linear change of variables $x \to Lx$ such that $B_1 \subset L(K) \subset nB_1$, where B_1 is the unit Euclidean ball and n is the dimension of space. Such a linear change of variables transforms equation (4.18) into an equation of the same type, so we can assume without loss of generality that $B_1 \subset K \subset nB_1$.

Then, classical arguments for second-order elliptic equations (comparison principles, barriers) show the existence of a universal constant C such that

$$\inf_K \varphi \ge -C, \qquad \varphi(x) \ge -C\, d(x, \partial K)^\alpha,$$

where $\alpha = 2/n$ if $n \ge 3$, $\alpha \in (0,1)$ if $n = 2$.

The next step is to get rid of the problem of degenerate convexity. This is the goal of the following lemma, due to Caffarelli:

Lemma 4.12 (Flatness is generated at the boundary). *Let φ satisfy (4.18) in some open, bounded convex set Ω, and let $A \subset \overline{\Omega}$ be the set of points where φ achieves its minimum value. Then, either A is reduced to a single point, or A has no extremal points in Ω.*

Some explanation for the title of the lemma is needed. The second possibility in the conclusion means that all extremal points of A in $\overline{\Omega}$ will lie on the boundary $\partial\Omega$. Of course, the same conclusion holds true if one replaces φ by $\varphi - \ell$, where ℓ is any affine function. Suppose that ℓ defines

a tangent hyperplane to the graph of φ; then the minimum of $\varphi - \ell$ is 0, and therefore A is just the set where the graph of φ coincides with this tangent. Then the conclusion of the lemma implies that the graph of φ cannot contain a flat part (i.e. a portion which coincides with a portion of a tangent hyperplane, and is not reduced to a single point), unless this flat part is "generated" at the boundary.

Now, consider again a solution to (4.18), in $\Omega = \{\varphi < 0\}$, so $\varphi = 0$ on $\partial\Omega$. Let x_0 be the point where φ achieves its minimum (there is a unique such point by the previous lemma). Let $C \subset \mathbb{R}^{n+1}$ be the cone of all lines emanating from $(x_0, \varphi(x_0))$ and passing through $(x, 0)$, where $x \in \partial\Omega$. All lines of C touch the graph of $\partial\varphi$ in two points, one of which is at the boundary of Ω (see Figure 4.2). Let $H_{1/2}$ be the horizontal hyperplane $(y = \varphi(x_0)/2)$; if $C \cap H_{1/2}$ touches the graph of φ, it follows by convexity that the graph of φ contains a whole line. This is however impossible, by a new application of Lemma 4.12.

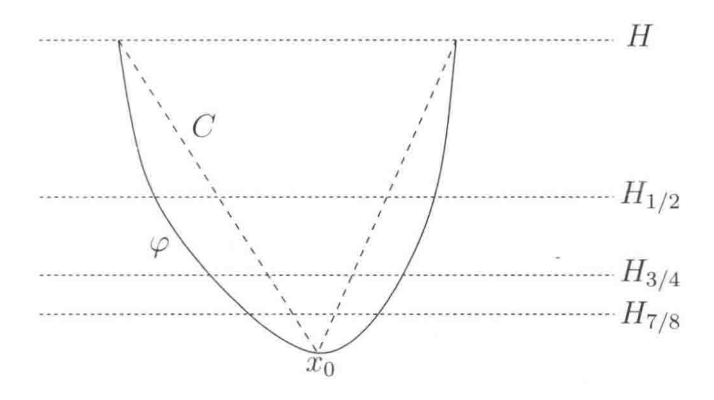


Figure 4.2. The graph of φ separates strictly from the cone C

As a consequence, there is a universal constant $D > 0$ such that the distance from $C \cap H_{1/2}$ to the graph of φ is bounded below by D. Indeed, if this is not the case, it is possible to construct a sequence of solutions φ^n to (4.18) whose graphs approach $C \cap H_{1/2}$ as $n \to \infty$. Since the functions φ^n are convex and uniformly bounded, they are also uniformly Lipschitz, and by Ascoli's theorem they define a relatively compact sequence for the topology of uniform convergence. So we can extract a subsequence $(n_k)_{k \in \mathbb{N}}$, such that φ^{n_k} converges uniformly to some convex function φ. Then, one can prove that φ is still a solution of (4.18): indeed, the Aleksandrov definition is stable under uniform convergence. On the other hand, the graph of φ should contain a point in the interior of $C \cap H_{1/2}$, and by convexity it should contain

a whole line segment. But Lemma 4.12 tells us that this is incompatible with φ solving (4.18).

By iteration of the argument and rescaling (consider $H_{3/4}$, $H_{7/8}$, etc., where $H_s = \{y = s\varphi(x_0)\}$), this property implies that the graph of φ looks "flatter and flatter" when approaching x_0. One can quantify this into an estimate that $\varphi \in C^{1,\alpha}$ at x_0 (see for instance [**80**]), for some $\alpha > 0$. Since φ is convex, any point can be transformed into a local minimum by just subtracting an affine function from φ; hence this argument shows that the whole of φ is of class $C^{1,\alpha}$.

After these first steps, one can attack the problem of second derivative estimates. The next theorem gathers some of the main results that were obtained by Caffarelli as an extension of the above methods.

Theorem 4.13 (Regularity for prescribed Hessian determinant). *Let φ be an Aleksandrov solution of*

$$\det D^2\varphi = h$$

in a bounded set Ω, $B_1 \subset \Omega \subset nB_1$. Then,

(i) If h is bounded from above and below by positive constants, then $\varphi \in C^{1,\alpha}(\Omega)$ for some universal exponent α.

(ii) Given any $\alpha > 0$, there exists $\varepsilon > 0$ such that if $1 - \varepsilon \leq h \leq 1 + \varepsilon$, then $\varphi \in C^{1,\alpha}(\Omega)$.

(iii) Given any $p \in (1, \infty)$, there exists $\varepsilon > 0$ such that if $1 - \varepsilon \leq h \leq 1 + \varepsilon$, then $\varphi \in W^{2,p}(\Omega)$. Also, if $h \in C(\Omega)$, then $\varphi \in W^{2,p}(\Omega)$ for all $p < \infty$.

(iv) If $h \in C^{0,\alpha}(\Omega)$, then $\varphi \in C^{2,\alpha}(\Omega)$.

As a further extension of these techniques, Caffarelli arrived at the following results.

Theorem 4.14 (Caffarelli's regularity theory for optimal transportation). *(i) Let $f, g \in C^{0,\alpha}$ $(0 < \alpha < 1)$ be Hölder-continuous functions defined on bounded open sets X and Y respectively, bounded from above and below by positive constants. Assume that Y is convex. Let φ be the unique Brenier solution of the Monge-Ampère equation*

$$\det D^2\varphi(x) = \frac{f(x)}{g(\nabla\varphi(x))}, \qquad x \in X.$$

Then, this equation is also satisfied in the Aleksandrov sense, and $\varphi \in C^{2,\alpha}(X)$. Moreover, if both X and Y are convex, then $\varphi \in C^{1,\alpha}(\overline{X})$.

(ii) If X and Y are uniformly convex and of class $C^{2,\alpha}$, $f \in C^{0,\alpha}(\overline{X})$, $g \in C^{0,\alpha}(\overline{Y})$, then $\varphi \in C^{2,\alpha}(\overline{X})$.

(iii) If $f, g \in C^{0,\alpha}(\mathbb{R}^n)$ are positive and bounded in the whole of $\mathbb{R}^n$, then $\varphi \in C^{2,\alpha}(\mathbb{R}^n)$.

Remarks 4.15. (i) The constants appearing in these various results are in fact uniform. For instance, point (i) could be strengthened as: for any relatively compact open subset O of Ω, the norm of φ in $C^{2,\alpha}(O)$ can be bounded by a constant which depends only on O, on upper bounds on the norms of f, g in $C^{0,\alpha}$, and on a pointwise lower bound on these functions; etc.

(ii) It is worth noting that once these $C^{2,\alpha}$ regularity estimates in the interior of X, Y have been proven, the classical theory of elliptic regularity can be used to prove, via bootstrapping arguments, that

$$f, g \in C^{k,\alpha} \Longrightarrow \varphi \in C^{k+2,\alpha}.$$

In particular, if f and g are C^∞ and positive, locally bounded from below on their respective supports, and the support of g is convex, then φ is itself of class C^∞.

Elements of proofs for part (iii) of the theorem (which is not explicitly stated in Caffarelli's papers) are presented in [**10**, Appendix].

4.3. Open problems

When this book was first published, the main open problem in the field was to derive regularity estimates for more general transportation costs. If the cost function is, say, of the form $c(|x - y|)$, one can always write down a Monge-Ampère type equation (see [**151**]), so it looks like a problem of regularity for fully nonlinear partial differential equations, just more complicated. But at the time, essentially nothing was known about the smoothness of the solutions to these equations, beyond the regularity properties which automatically follow from c-concavity. On closer examination, the problems seemed even more tricky than some of the outstanding open problems in the classical theory of the Monge-Ampère equation, which have been out there for decades. The same remarks held true for transportation on a Riemannian manifold, even for a quadratic cost.

While modest progress was slowly achieved in this direction after publication of the book, it was a major blow when Ma, Trudinger and Wang [**196**] discovered, by purely analytical means, a fourth-order structure condition on the cost function, from which regularity property could be derived. Loeper [**195**] further gave geometric interpretations of this condition, and continuity theorems in lower regularity as well; but most importantly, he established that the seemingly intricate condition of Ma, Trudinger and Wang was close to be necessary.

An enormous array of work has followed, leading to certain disturbing conclusions and showing that regularity is in general a rare property in the world of optimal transportation. For instance [**140**], in a Riemannian setting, with the quadratic cost function, regularity may drastically fail even when the densities are smooth and positive, even if the manifold is an ellipsoid! On the other hand, regularity does hold if the manifold is a round sphere. We refrain from saying more here, or even from writing down the complicated Ma-Trudinger-Wang condition, and refer to [**267**, Chapter 12] for a detailed introduction.

A much more modest goal was to focus on the Euclidean situation and obtain constructive estimates for Caffarelli's theory. Since some of the crucial bounds there were obtained by a compactness argument, these works say nothing about the explicit size of the constants; this problem was obviously related to the issue of quantifying more precisely the strict convexity of solutions. This question, which was presented as an open problem in the first edition of this book, has been solved by Forzani and Maldonado [**146**].

Chapter 5

Displacement Interpolation and Displacement Convexity

Up to now we have browsed through most of the theory of optimal transference plans. In the present chapter we pursue several goals. First, in Sections 5.1 and 5.2 we shall introduce the notions of displacement interpolation and displacement convexity. Despite their simplicity, these concepts have powerful applications outside the field of optimal transportation, and from the beginning we shall present them a little bit from a physicist's perspective. In Section 5.3 we shall discover some of their applications; further ones will be found in the next chapters. Finally, in Section 5.4 we introduce an appealing interpretation of optimal transportation in the language of fluid mechanics and Hamilton-Jacobi equations. The present chapter is crucial for a thorough understanding of Chapters 6, 8 and 9.

5.1. Displacement interpolation

5.1.1. Time-dependent Monge-Kantorovich problem. So far we have only considered a time-independent minimization problem: the cost function for transporting one unit of mass from one location to another was a function of the initial and the final locations, but did not depend on the actual history of the transportation. Now we shall change our model to take this history into account.

Monge himself thought it necessary to formulate the optimal transportation problem in terms of trajectories, for practical purposes (for instance, to take obstacles into account) and for theoretical purposes as well (for a distance cost function, the fact that trajectories cannot cross is a powerful tool in the investigation of the problem). And after all, to achieve a transport process it would be better to know the trajectories of all particles. But the mathematical importance of the time variable was put forward by Benamou and Brenier [**39**], in the case of a quadratic cost function. Roughly speaking, if the mass transportation problem is viewed as a *distance* problem (find the distance between a probability measure μ and a probability measure ν), then the time-dependent minimization problem can be viewed as a *geodesic* problem (find an optimal path between μ and ν).

As we shall see, most of the hard work has already been done, and in many cases of interest the solution of the time-dependent transportation problem is actually contained in the solution of the time-independent transportation problem. To be honest, we should add that the time-dependent transportation problem is usually much more difficult when this reduction to a time-independent problem cannot be performed.

Recall the classical formulation of the Monge problem,

$$\inf\left\{\int_X c(x,T(x))\,d\mu(x);\quad T\#\mu=\nu\right\}. \tag{5.1}$$

In our new modelling we shall study a transportation process via the family of the trajectories of all points: to each x will be associated a trajectory $(T_t(x))_{0\le t\le 1}$, which we shall often abbreviate as (T_tx), and $C[(T_tx)]$ will be the corresponding displacement cost. We shall require $t\longmapsto T_tx$ to be continuous and piecewise C^1 for $d\mu$-almost all x, and we shall look for the solution of the **time-dependent minimization problem**

$$\inf\left\{\int_X C\big[(T_tx)_{0\le t\le 1}\big]\,d\mu(x);\qquad T_0=\mathrm{Id}\,,\quad T_1\#\mu=\nu\right\}. \tag{5.2}$$

Problems (5.1) and (5.2) are **compatible** if they predict the same total cost and the same displacement map, in the sense that each optimal (T_t) in (5.2) gives rise to an optimal T in (5.1), via $T=T_1$. A simple and natural sufficient condition for compatibility is that for all x and y,

$$c(x,y)=\inf\big\{C\big[(z_t)_{0\le t\le 1}\big];\quad z_0=x,\quad z_1=y\big\}. \tag{5.3}$$

If the problem is set on a differentiable structure, and $\dot z_t$ denotes the derivative of z_t with respect to t, then many cases of interest arise with $C[(z_t)]=\int_0^1 c(\dot z_t)\,dt$, in which case $c(z)$ can be seen as a **differential cost**.

Examples 5.1.

$$C[(z_t)] = \int_0^1 |\dot{z}_t|^2\,dt \quad \text{in } \mathbb{R}^n \qquad \Longrightarrow c(x,y) = |x-y|^2.$$

$$C[(z_t)] = \int_0^1 |\dot{z}_t|^p\,dt, \quad p \geq 1, \quad \text{in } \mathbb{R}^n \qquad \Longrightarrow c(x,y) = |x-y|^p.$$

$$C[(z_t)] = \int_0^1 \|\dot{z}_t\|^p\,dt \quad \text{on a smooth complete Riemannian manifold } M$$
$$\Longrightarrow c(x,y) = d(x,y)^p.$$

The first two examples are actually particular cases of the elementary

Proposition 5.2 (Extremal trajectories for convex costs are straight lines). *If c is a convex function on $\mathbb{R}^n$, then*

$$\inf\left\{\int_0^1 c(\dot{z}_t)\,dt; \quad z_0 = x,\ z_1 = y\right\} = c(y-x). \tag{5.4}$$

Variant: given any $T > 0$,

$$\inf\left\{\int_0^T c(\dot{z}_t)\,dt; \quad z_0 = x,\ z_T = y\right\} = Tc\left(\frac{y-x}{T}\right). \tag{5.5}$$

Moreover, if c is strictly convex, then the infimum is achieved uniquely by

$$z_t = x + t(y-x) \qquad \left(\textit{variant}: x + \frac{t}{T}(y-x)\right).$$

The proof is an easy consequence of Jensen's inequality for convex functions, and is left as an exercise.

Important Remarks 5.3. (i) If $c(z) = |z|^p$ $(p \geq 1)$ on $\mathbb{R}^n$, then the infima in (5.4) and (5.5) are the same up to a multiplicative factor which depends only on T. A similar remark holds in the case of a Riemannian manifold.

(ii) In the situation of Proposition 5.2 for a strictly convex differential cost c, the only optimal trajectories are straight lines, parametrized with constant velocity. Similarly, the only optimal trajectories for the differential cost $c(z) = \|z\|^p$ on a manifold $(p > 1)$ are the **minimizing geodesics** with **arc length parametrization**. They may be nonunique. On the contrary, the case $p = 1$ is degenerate in the sense that any time-reparametrization of a minimizing geodesic is a minimizer.

Of course, various relaxed versions of this minimization problem can be formulated, in the same spirit as the Kantorovich relaxation of the Monge problem.

If condition (5.3) is enforced, then solutions of the time-dependent minimization problem have to satisfy the following condition: for $d\mu$-almost all x, $(T_t x)_{0\le t\le 1}$ is optimal in (5.3), i.e.

$$c(x, T(x)) = C\big[(T_t x)_{0\le t\le 1}\big].$$

In words, up to a negligible set of initial locations, each trajectory should be optimal.

Examples 5.4. If $c(x,y) = c(x-y)$ in $\mathbb{R}^n$, with c strictly convex, $c(0) = 0$, then almost all trajectories have to be straight lines. If $c(x,y) = d(x,y)^p$ $(p \ge 1)$ on M, then almost all trajectories have to be minimizing geodesics.

The aforementioned considerations may be sufficient to solve the time-dependent minimization problem starting from known results for the time-independent minimization problem. In particular we have the following result as a consequence of Theorem 2.44.

Theorem 5.5 (Time-dependent optimal transportation theorem). *Consider the cost function $c(x,y) = c(x-y)$ in $\mathbb{R}^n$, with c strictly convex, $c(0) = 0$. Let μ, ν be two probability measures in $\mathbb{R}^n$, absolutely continuous with respect to Lebesgue measure, and let $C[(z_t)] = \int_0^1 c(\dot{z}_t)\, dt$. Let $\nabla\psi$ be the ($d\mu$-almost everywhere) unique gradient of a c-concave function ψ such that $[\mathrm{Id} - \nabla c^*(\nabla\psi)]\#\mu = \nu$. Then the solution of problem* (5.2) *is given by*

$$T_t(x) = x - t\nabla c^*(\nabla\psi(x)), \qquad 0 \le t \le 1. \tag{5.6}$$

Proof. If it exists, an optimizer has to satisfy $T_0(x) = x$, $T_1(x) = x - \nabla c^*(\nabla\psi(x))$, and $T_t(x)$ should interpolate linearly between $T_0(x)$ and $T_1(x)$. This leads to equation (5.6). Then it is easy to check that this is indeed a minimizer (exercise). □

If we now consider transportation on a Riemannian manifold M, with cost $d(x,y)^2/2$, then we know by Theorem 2.47 that the optimal transportation is of the form $T(x) = \exp_x(-\nabla\psi(x))$, so that we would expect the solution of the time-dependent minimization problem to be given by the geodesic path

$$T_t(x) = \exp_x\big(-t\nabla\psi(x)\big). \tag{5.7}$$

However, this will be true only if we know that this geodesic is *minimizing*, in other words (in the language of differential geometry) if we have not gone past the **cut locus** in the transportation process. Recall that the cut locus associated with a point $x \in M$ is the family $\mathcal{C}(x)$ of all points of the form

γ_{t_0}, where $(\gamma_t)_{t\geq 0}$ is a geodesic which is minimizing between γ_0 and γ_t for $t \leq t_0$, but not for $t > t_0$.

It is indeed the case that the geodesic path (5.7) remains minimizing, as we shall see in the sequel.

5.1.2. The homogeneous case. By definition, we say that a cost function is homogeneous if it can be written as $c(x, y) = |x - y|^p$ in $\mathbb{R}^n$, or $c(x, y) = d(x, y)^p$ on a smooth complete manifold; we shall only consider $p \geq 1$. For such costs, the solution of the time-dependent minimization problem is invariant under reparametrization of time, as pointed out in Remark 5.3 (ii). A tedious but simple argument exploiting this remark leads to the following proposition. *Let μ, ν be absolutely continuous probability measures in $\mathbb{R}^n$ or on a smooth complete Riemannian manifold, and let $c(x, y)$ be a homogeneous cost function; assume that there exists a unique solution to the Monge minimization problem. Then $(T_t)_{0\leq t\leq 1}$ is a solution of the time-dependent minimization problem* (5.2) *if and only if, for all intermediate times $t_0 \in [0, 1]$, the family $(T_t)_{0\leq t\leq t_0}$ yields an optimal transportation from μ to $T_{t_0}\#\mu$, and the family $(T_t)_{t_0\leq t\leq 1}$ yields an optimal transportation from $T_{t_0}\#\mu$ to ν.*

The proof would be routine, were it not for the possibility of trajectories crossing at time t_0, in which case $(T_t)_{t_0\leq t\leq 1}$ should be interpreted in a generalized sense. Anyway we shall not need this proposition later on, and it is stated only for the sake of clarity. For the present purposes, we shall be content with the following theorem.

Theorem 5.6 (Intermediate time optimality theorem). *Consider the solution of the Monge-Kantorovich problem in the following cases:*

(i) μ, ν do not give mass to small sets, $c(x-y) = |x-y|^p$ in $\mathbb{R}^n$ $(p > 1)$, and the optimal transportation takes the form $T(x) = x - \nabla c^*(\nabla\psi(x))$, where ψ is given by the Gangbo-McCann theorem (Theorem 2.45*);*

(ii) μ, ν are absolutely continuous and compactly supported in a smooth complete Riemannian manifold M, $c(x, y) = d(x, y)^2$, and the optimal transportation takes the form $T(x) = \exp_x(-\nabla\psi(x))$, where ψ is given by McCann's theorem (Theorem 2.47*);*

and for all $t \in [0, 1]$ define T_t by changing ψ for $t\psi$ in the expression of T. Then, for all $t \in [0, 1]$, T_t is also optimal in the transportation from μ to $T_t\#\mu$.

Proof. Consider for instance case (ii). Recall that $d^2/2$-concave functions are functions of the form

$$\psi(x) = \inf_{y\in M}\left[\frac{d(y, x)^2}{2} + \zeta(y)\right], \qquad \zeta : M \to \mathbb{R} \cup \{-\infty\}. \tag{5.8}$$

What we have to prove is that $t\psi$ is also $d^2/2$-concave whenever $0 \le t \le 1$. From formula (5.8) we see that we just have to treat the particular case when $\psi(x) = d(z,x)^2/2$ for some fixed $z \in M$. So, all we need to prove is that whenever $\lambda \in [0,1]$, one can write

$$\lambda\, d(z,x)^2 = \inf_{y \in M} \left[\frac{d(y,x)^2}{2} + \zeta(y) \right]. \tag{5.9}$$

But this is just a particular case of a well-known identity,

$$\inf_y \left[\frac{d(x,y)^2}{a} + \frac{d(y,z)^2}{b} \right] = \frac{d(x,z)^2}{a+b}, \qquad a,b > 0. \tag{5.10}$$

To deduce (5.9) from (5.10) it is sufficient to choose a and b in such a way that $a/(a+b) = \lambda$. □

Exercise 5.7. Prove formula (5.10). Prove case (i) of Theorem 5.6 by a similar method.

Remark 5.8. The above argument is due to Cordero-Erausquin, McCann and Schmuckenschläger [**100**]. It may be illuminating for readers with a partial-differential-equations oriented mind to notice that equation (5.10) is nothing but a Hopf-Lax formula (see subsection 5.4.6 below). So what we have just proven is strongly linked to the following invariance property of the Hamilton-Jacobi equation: if u solves

$$\frac{\partial u}{\partial t} + c^*(\nabla u) = 0,$$

where $c(z) = |z|^p$, then $\lambda u(\lambda^{p'-1}t, \cdot)$ is also a solution of the same equation ($p' = p/(p-1)$). We shall come back briefly to this remark in subsection 5.4.6.

5.1.3. McCann's interpolation. A particularly important case of the preceding discussion occurs when $c(x,y) = |x-y|^2$ in $\mathbb{R}^n$. Then the solution of the time-dependent minimization problem coincides with *McCann's interpolation* [**203**], which we shall often call (following McCann) **displacement interpolation**. Let us give a few details on the properties of this interpolation.

Let μ, ν be two probability measures on $\mathbb{R}^n$. Assume that μ, ν do not give mass to small sets. According to Theorem 2.32 there exists a ($d\mu$-almost everywhere unique) gradient of a convex function $\nabla\varphi$ such that $\nabla\varphi\#\mu = \nu$. Define

$$\rho_t = [\mu,\nu]_t \equiv \big[(1-t)\mathrm{Id} + t\nabla\varphi\big]\#\mu. \tag{5.11}$$

The family of probability measures $(\rho_t)_{0 \le t \le 1}$ interpolates between μ and ν in a remarkable way. Of course,

$$[\mu,\nu]_0 = \mu, \qquad [\mu,\nu]_1 = \nu.$$

Moreover, since $(1-t)\mathrm{Id} + t\nabla\varphi = \nabla\big[(1-t)|\cdot|^2/2 + t\varphi\big]$ is always the gradient of a convex function, we see by Theorem 2.12 that the value of the optimal cost for transporting μ onto $[\mu,\nu]_t$ is

$$\begin{aligned}\mathcal{T}_2(\mu,\rho_t) &= \int_{\mathbb{R}^n} \Big| x - \big[(1-t)x + t\nabla\varphi(x)\big]\Big|^2 \, d\mu(x) \\ &= t^2 \int_{\mathbb{R}^n} |x - \nabla\varphi(x)|^2 \, d\mu(x) = t^2 \mathcal{T}_2(\mu,\nu).\end{aligned}$$

In terms of the quadratic Wasserstein distance $W_2 = \sqrt{\mathcal{T}_2}$,

$$W_2(\mu,\rho_t) = t\, W_2(\mu,\nu). \tag{5.12}$$

Other elementary properties of McCann's interpolation are gathered below.

Proposition 5.9 (Basic properties of displacement interpolation). *With the same notation as above, one has*

(i) $[\mu,\nu]_t = [\nu,\mu]_{1-t}$;

(ii) $\big[[\mu,\nu]_t, [\mu,\nu]_{t'}\big]_s = [\mu,\nu]_{(1-s)t+st'}$;

(iii) if μ or ν is absolutely continuous, then so is $[\mu,\nu]_t$, for all $t \in (0,1)$.

Proof. To prove (i), we just have to note that

$$\begin{aligned}[\mu,\nu]_t &= \big((1-t)\,\mathrm{Id} + t\nabla\varphi\big)\#\mu \\ &= \big((1-t)\,\mathrm{Id} + t\nabla\varphi\big)\#(\nabla\varphi^*\#\nu) \\ &= \Big[\big((1-t)\,\mathrm{Id} + t\nabla\varphi\big)\circ\nabla\varphi^*\Big]\#\nu \\ &= \big((1-t)\,\nabla\varphi^* + t\,\mathrm{Id}\,\big)\#\nu.\end{aligned}$$

The proof of (ii) is just an easy computation, left as an exercise.

We turn to the proof of (iii). Thanks to (i) we just have to consider the case in which μ is absolutely continuous. We define

$$\varphi_t(x) = t\varphi(x) + (1-t)\frac{|x|^2}{2},$$

and we note that

$$\big\langle \nabla\varphi_t(x) - \nabla\varphi_t(y),\, x-y\big\rangle \geq (1-t)|x-y|^2.$$

So in particular

$$|\nabla\varphi_t(x) - \nabla\varphi_t(y)| \geq (1-t)|x-y|. \tag{5.13}$$

Since φ_t is uniformly convex, its Legendre transform φ_t^* is everywhere differentiable, and from (5.13) we deduce that $\nabla\varphi_t^* = (\nabla\varphi_t)^{-1}$ is Lipschitz with Lipschitz norm less than $(1-t)^{-1}$. In particular, whenever A is of zero

Lebesgue measure, then $\nabla\varphi_t^*(A)$ is also of zero Lebesgue measure. Then we can write, using Lemma 4.6 from Chapter 4,

$$\rho_t[A] = \mu[\partial\varphi_t^*(A)] = \mu[\nabla\varphi_t^*(A)] = 0.$$

□

We shall see in the next section some of the advantages of this interpolation with respect to the standard "linear" interpolation $\rho_t = (1-t)\mu + t\nu$, and also how these two interpolations can be put in a common framework.

5.2. Displacement convexity

Whenever ρ is an absolutely continuous probability measure on $\mathbb{R}^n$, we shall identify it with its Lebesgue density, and write $d\rho(x) = \rho(x)\,dx$. We shall denote by $P_{\rm ac}(\mathbb{R}^n)$ the set of all such probability measures.

Let $\mu, \nu \in P_{\rm ac}(\mathbb{R}^n)$. Introduce the displacement interpolation $(\rho_t)_{0\le t\le 1}$ between μ and ν, associated with the time-dependent minimization problem for a quadratic cost function. Recall from the last section the formula

$$\rho_t = [(1-t)\mathrm{Id} + t\nabla\varphi]\#\mu, \qquad 0 \le t \le 1,$$

with φ convex, and $\rho_1 = \nu$. More generally, for a strictly convex cost function c, one can define

$$\rho_t = [\mathrm{Id} - t\nabla c^*(\nabla\psi)]\#\mu, \qquad 0 \le t \le 1,$$

where ψ is c-concave and $\rho_1 = \nu$.

5.2.1. Definitions. A natural question is the following. If F is a functional on the space of probability measures, what can be said about the behavior of $F(\rho_t)$ as t varies in $[0,1]$? Among properties to study, convexity is of course in the first place. This motivates the following definition.

Definition 5.10 (Displacement convexity). (i) A subset $\mathcal{P}$ of $P_{\rm ac}(\mathbb{R}^n)$ is said to be displacement convex if it is stable under displacement interpolation: for all μ, ν in $\mathcal{P}$, and for all $t \in [0,1]$, the displacement interpolant $\rho_t = [\mu, \nu]_t$ still lies in $\mathcal{P}$.

(ii) Let F be a functional defined on a displacement convex subset $\mathcal{P}$ of $P_{\rm ac}(\mathbb{R}^n)$, with values in $\mathbb{R} \cup \{+\infty\}$. It is said to be *displacement convex on* $\mathcal{P}$ if the following property holds: whenever $\rho_0 = \mu$ and $\rho_1 = \nu$ are given elements of $\mathcal{P}$ and $(\rho_t)_{0\le t\le 1}$ is their displacement interpolant, then $t \mapsto F(\rho_t)$ is convex on $[0,1]$.

Remark 5.11. It is important to note that the classical definition of convexity is just the same, except that the displacement interpolation is replaced by the *linear interpolation*,

$$\widetilde{\rho}_t = (1-t)\mu + t\nu.$$

In some situations, one may wish to extend this definition to more general probability measures, possibly singular. Here is an alternative definition, which works also for singular measures. However, since it is explicitly based on the optimal transportation problem with quadratic cost, we restrict this definition to the set of measures in $P_2(\mathbb{R}^n)$, to make sure that optimal transference plans always exist. This definition can also be generalized to other cost functions.

Definition 5.12 (Displacement convexity again). Let $\sigma_t : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ be defined by $\sigma_t(x,y) = (1-t)x + ty$.

(i) A subset $\mathcal{P}$ of $P_2(\mathbb{R}^n)$ is said to be displacement convex if for all $\rho_0, \rho_1 \in \mathcal{P}$, for all π optimal in the Monge-Kantorovich problem with probability measures ρ_0 and ρ_1 and quadratic cost, and for all $t \in [0,1]$, $\rho_t := \sigma_t \# \pi$ lies in $\mathcal{P}$.

(ii) With the same notation, a functional F on $\mathcal{P}$ is said to be displacement convex if for all ρ_0, ρ_1 in $\mathcal{P}$,

$$t \longmapsto F(\rho_t) \quad \text{is convex on } [0,1].$$

(iii) For any strictly convex cost function $c(x-y)$ one can define the concept of *c-displacement convexity* in a similar way, provided that $\mathcal{T}_c(\mu,\nu) < +\infty$ for all $\mu, \nu \in \mathcal{P}$, choosing π in (i) as an optimizer in the Monge-Kantorovich problem with cost function c.

Remarks 5.13. (i) It is easy to check that Definitions 5.10 and 5.12 agree on $P_{\mathrm{ac},2}(\mathbb{R}^n) = P_{\mathrm{ac}} \cap P_2(\mathbb{R}^n)$, and it also easy to check that $P_{\mathrm{ac},2}(\mathbb{R}^n)$ is displacement convex.

(ii) In almost all cases of applications known so far, it is not necessary to use displacement convexity for non-quadratic costs. An interesting exception was encountered by Cullen and Maroofi [**107**] in their study of the compressible semi-geostrophic system. In the sequel, we only consider quadratic costs.

Further variants of Definition 5.12 can be given. For instance,

Definition 5.14 (Variants of displacement convexity). With the same notation as in Definition 5.12, a functional F is said to be *strictly* displacement convex on $\mathcal{P}$ if

$$\forall \rho_0, \rho_1 \in \mathcal{P}, \qquad \rho_0 \neq \rho_1 \quad \Longrightarrow \quad \Big[t \longmapsto F(\rho_t) \text{ is strictly convex on } [0,1] \Big].$$

It is said to be λ-*uniformly displacement convex* on $\mathcal{P}$ for some $\lambda > 0$ if for all $\rho_0, \rho_1 \in \mathcal{P}$,

$$\frac{d^2}{dt^2} F(\rho_t) \geq \lambda\, \mathcal{T}_2(\rho_0, \rho_1), \qquad t \in (0,1).$$

It is said to be *semi-displacement-convex* on $\mathcal{P}$, with constant $C \geq 0$, if for all $\rho_0, \rho_1 \in \mathcal{P}$,

$$\frac{d^2}{dt^2} F(\rho_t) \geq -C\,\mathcal{T}_2(\rho_0, \rho_1), \qquad t \in (0,1).$$

Recall that $\mathcal{T}_2(\mu, \nu)$ is a shorthand for the total transportation cost between μ and ν with quadratic cost function. Similar definitions can be given for non-quadratic cost functions, mutatis mutandis; or for probability measures defined on a Riemannian manifold.

In the sequel, we shall study the displacement convexity of certain functionals which present a physical interest. For simplicity, we restrict all the discussion to the Euclidean case, and it is only in subsection 5.2.7 that we shall make a few remarks about the more general case of a Riemannian manifold.

5.2.2. The three basic examples. We shall now study the following three typical functionals from the point of view of displacement convexity. Remember that we identify an absolutely continuous probability measure ρ with its density.

- **internal energy:**

$$\mathcal{U}(\rho) = \int_{\mathbb{R}^n} U\big(\rho(x)\big)\, dx, \qquad U \text{ measurable } \mathbb{R}_+ \to \mathbb{R} \cup \{+\infty\}; \tag{5.14}$$

- **potential energy:**

$$\mathcal{V}(\rho) = \int_{\mathbb{R}^n} V\, d\rho, \qquad V \text{ measurable } \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}; \tag{5.15}$$

- **interaction energy:**

$$\mathcal{W}(\rho) = \frac{1}{2} \int_{\mathbb{R}^n \times \mathbb{R}^n} W(x-y)\, d\rho(x)\, d\rho(y), \qquad W \text{ measurable } \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}. \tag{5.16}$$

The function U may be called a **density of internal energy**, the function V a **potential** and the functional W an **interaction potential**. The domains of definition of $\mathcal{U}$, $\mathcal{V}$ and $\mathcal{W}$ will depend on the behavior of U, V and W, respectively. For instance,

- $\mathcal{U}$ is well-defined on $P_{\mathrm{ac}}(\mathbb{R}^n)$, with values in $\mathbb{R} \cup \{+\infty\}$, as soon as $U \geq 0$. It is not identically $+\infty$ as soon as $U(0) = 0$ and U is not identically $+\infty$ on $\mathbb{R}_+ \setminus \{0\}$.
- $\mathcal{V}$ (resp. $\mathcal{W}$) is well-defined on $P(\mathbb{R}^n)$, with values in $\mathbb{R} \cup \{+\infty\}$, as soon as V (resp. W) is bounded below by some real number.

The restriction about U being nonnegative is sometimes too strong. In many situations one can define $\mathcal{U}$ unambiguously even if U takes negative values; see Remark 5.16 (ii) below. We also note that U is in practice continuous.

The following important theorem is mainly due to McCann [**203**].

Theorem 5.15 (Criteria for displacement convexity). *Let $\mathcal{P}$ be a displacement convex subset of either $P_{ac}(\mathbb{R}^n)$ (for point (i)), or $P_2(\mathbb{R}^n)$ (for points (ii) and (iii)), on which $\mathcal{U}$, $\mathcal{V}$ and $\mathcal{W}$ are well-defined with values in $\mathbb{R} \cup \{+\infty\}$. Then,*

(i) If U satisfies $U(0) = 0$ and

$$\Psi : r \longmapsto r^n U(r^{-n}) \qquad \textit{is convex nonincreasing on } (0, +\infty), \tag{5.17}$$

then $\mathcal{U}$ is displacement convex on $\mathcal{P}$. Conversely, if Ψ is nonincreasing and $\mathcal{U}$ is displacement convex on $P_{ac}(\mathbb{R}^n)$, then Ψ is convex.

(ii) If V is convex (resp. strictly convex, λ-uniformly convex, semiconvex with constant C), then $\mathcal{V}$ is displacement convex (resp. strictly displacement convex, λ-uniformly displacement convex, semi-displacement-convex with constant C) on $\mathcal{P}$. Conversely, if $\mathcal{V}$ is displacement convex on $P_2(\mathbb{R}^n)$, then V is convex.

(iii) If W is convex (resp. semiconvex with constant C), then $\mathcal{W}$ is displacement convex (resp. semi-displacement-convex with constant C). If W is strictly convex (resp. λ-uniformly convex), then, for all $m \in \mathbb{R}^n$, $\mathcal{W}$ is strictly displacement convex (resp. λ-uniformly displacement convex) on the subspace $\mathcal{P}_m$ of probability measures in $\mathcal{P}$ having mean m. As a partial converse, if W is even continuous, and $n \geq 2$, and $\mathcal{W}$ is displacement convex on $P_2(\mathbb{R}^n)$, then W is convex.

Remarks 5.16. (i) The mean, or **center of mass**, of a probability measure ρ is $\int x\, d\rho(x)$. Note that $\mathcal{P}_m$ in (iii) is a displacement convex set (why?).

(ii) The set $\mathcal{P}$ can often be taken to be the whole of $P_{ac}(\mathbb{R}^n)$, i.e. all absolutely continuous probability measures. This is the case if the sufficient conditions given before Theorem 5.15 are fulfilled. However, in some situations of interest, one may be led to work on a smaller set. An example which arises in physics is $U(\rho) = \rho \log \rho$: then $\mathcal{U}$ takes both values $+\infty$ and $-\infty$ on $P_{ac}(\mathbb{R}^n)$. But if one defines for instance $\mathcal{P} = P_{ac,2}(\mathbb{R}^n) \equiv P_{ac}(\mathbb{R}^n) \cap P_2(\mathbb{R}^n)$, then $\mathcal{U}$ can be defined unambiguously on $\mathcal{P}$, with values in $\mathbb{R} \cup \{+\infty\}$, as can be seen by choosing $\varphi(x) = |x|^2$ in the inequality

$$\int \rho \log \rho \, d\nu + \int \varphi \rho \, d\nu \geq -\log \left(\int e^{-\varphi} \, d\nu \right),$$

which holds true whenever ν is a probability measure and ρ a probability density with respect to ν.

(iii) In Theorem 5.15, one may also replace the property of displacement convexity by that of c-displacement convexity, where c is any strictly convex superlinear cost function, in the sense of Theorem 2.44. But actually, the displacement convexity property of $\mathcal{W}$ turns out not to depend on c.

(iv) More complicated interaction energies could also be considered, like

$$\int W\big(L(x_1,\ldots,x_k)\big)\,d\rho(x_1)\,\ldots\,d\rho(x_k),$$

where L is an arbitrary linear function.

The next three subsections are devoted to the proof of Theorem 5.15. The proof of (i) will be based on Theorem 4.8, while the proofs of (ii) and (iii) will be more direct. Before going into them, we mention the following open problem, which has not seen much progress since the first edition of this book:

Open Problem 5.17 (Are there other interesting displacement convex functionals?). *Besides the three examples stated there, can one find other useful examples of displacement convex functionals? In particular, functionals involving gradients of ρ?*

5.2.3. Internal energy. Before we prove the first part of Theorem 5.15, a few remarks are in order concerning condition (5.17):

Remarks 5.18. (i) First of all, let us explain a little bit about the physical meaning of (5.17). Consider a homogeneous (or uniform) cloud of n-dimensional gas with mass M in a volume V, so that its density is constant and equal to M/V. Assume that the gas expands: its dimensions are multiplied by a factor λ, so its volume is multiplied by a factor λ^n, and its density is divided by a factor λ^n. So the internal energy of the gas, as a function of the dilation factor λ, is $V\lambda^n U(\lambda^{-n}M/V)$, which is proportional to $r^n U(r^{-n})$ if $r = \lambda(V/M)^{1/n}$. In other words, condition (5.17) means that the internal energy is a *convex* nonincreasing function of this factor. Note that physical realism requires at least that the internal energy be a nonincreasing function.

(ii) Let us assume that U is differentiable. It is often convenient to translate condition (5.17) in terms of the **thermodynamical pressure**,

$$P(\rho) = \rho U'(\rho) - U(\rho). \tag{5.18}$$

Let us briefly explain formula (5.18). Consider again a homogeneous cloud of gas with mass M, volume V and density $\rho = M/V$, and remember from

basic thermodynamics that the pressure P may be defined by the formula

$$P(\rho) = -\frac{d\mathcal{U}}{dV}$$

(in a more common formulation, the infinitesimal work dW which is necessary for imposing a change of volume dV is given by $-PdV$). It is reasonable to assume that the gas has zero internal energy if it is dilute in the whole space ($V = \infty$). Hence, if V_0 is a given volume, and $\rho_0 = M/V_0$, one can write

$$U(\rho_0)V_0 = \int_{V_0}^{\infty} \left(-\frac{d\mathcal{U}}{dV}\right) dV = \int_{V_0}^{\infty} P\left(\frac{M}{V}\right) dV = M \int_0^{\rho_0} P(\rho)\,\frac{d\rho}{\rho^2},$$

so that

$$U(\rho_0) = \rho_0 \int_0^{\rho_0} \frac{P(\rho)}{\rho^2}\, d\rho.$$

Formula (5.18) follows immediately from this identity. Note that this can make sense only if $P(0) = 0$.

(iii) The first derivative of $\Psi : r \longmapsto r^n U(r^{-n})$ is $-nr^{n-1}P(r^{-n})$, so the nonincreasing property of $r^n U(r^{-n})$ is equivalent to the nonnegativity of P, which makes physical sense. As for the second derivative of Ψ, it is given by

$$\Psi''(r) = n^2 r^{n-2}[r^{-n}P'(r^{-n}) - (1 - 1/n)P(r^{-n})];$$

hence the convexity of Ψ is equivalent to

$$\rho P'(\rho) \geq \left(1 - \frac{1}{n}\right) P(\rho). \tag{5.19}$$

This condition can be reformulated into the following **pressure condition for displacement convexity**:

$$\rho \longmapsto \frac{P(\rho)}{\rho^{1-\frac{1}{n}}} \qquad \text{is nondecreasing.} \tag{5.20}$$

(iv) In particular, since $P(0) = 0$, P should always be nonnegative, and from (5.19) it should be nondecreasing; since $P'(\rho) = \rho U''(\rho)$, it follows that U has to be convex. Thus, *for a realistic functional of the form of $\mathcal{U}$, displacement convexity is a stronger assumption than plain convexity.*

Examples 5.19. Here are some of the most typical energy densities satisfying condition (5.20):

$U(\rho) = \rho^\gamma$, $\gamma \geq 1$, in which case $P(\rho) = (\gamma - 1)\rho^\gamma$;

$U(\rho) = \rho \log \rho$, in which case $P(\rho) = \rho$;

$U(\rho) = -\rho^\gamma$, $(1 - 1/n) \leq \gamma \leq 1$, in which case $P(\rho) = (1 - \gamma)\rho^\gamma$.

Remark 5.20. A famous particular case is $U(\rho) = \rho^{5/3}$ in dimension $n = 3$, in which case $\mathcal{U}$ is the semiclassical approximation to the quantum kinetic energy of a three-dimensional gas of fermions, see Lieb [**187**].

Proof of part (i) of Theorem 5.15. Let U satisfy the assumption (5.17), and let μ, ν be two probability densities. We consider McCann's interpolation, in the form

$$\rho_t = (\mathrm{Id} - t\theta)\#\mu,$$

where θ is of the form $(\mathrm{Id} - \nabla\varphi)$, and φ is convex. We shall denote by $\nabla\theta$ the Jacobian matrix of θ.

As a consequence of Theorem 4.8 (iv), we have

$$\mathcal{U}(\rho_t) = \int_{\mathbb{R}^n} U\left(\frac{\rho(x)}{\det(I_n - t\nabla\theta(x))}\right)\det(I_n - t\nabla\theta(x))\,dx. \tag{5.21}$$

Next note that, as a function of t, the integrand in the right-hand side of (5.21) is given by the composition of the two mappings

$$\begin{cases} t \longmapsto \lambda = \det(I_n - tS)^{1/n}, \\ \\ \lambda \longmapsto U\left(\frac{r}{\lambda^n}\right)\lambda^n, \end{cases} \tag{5.22}$$

where $r = \rho(x)$ and $S = \nabla\theta(x)$ is a symmetric matrix, $S \leq I_n$.

The behavior of λ as a function of t is described by the following lemma, to be proven later.

Lemma 5.21 (Concavity of $\det^{1/n}$). *Given a symmetric matrix $S \leq I_n$, the function $t \longmapsto \det(I_n - tS)^{1/n}$ is concave (strictly unless S is a multiple of the identity).*

With formula (5.21) and Lemma 5.21 in hand, one can immediately conclude that Ψ is convex. Indeed, the two functions appearing in (5.22) are respectively convex nonincreasing and concave, so their composition is convex (exercise) with respect to t. □

Exercise 5.22 (Necessary condition for displacement convexity of $\mathcal{U}$). Complete the proof of (i): show that if $\mathcal{U}$ is displacement convex on $P_{\mathrm{ac}}(\mathbb{R}^n)$, then Ψ is convex.

Hint: Use the heuristics described above about the expanding cloud of gas.

Lemma 5.21 is a manifestation of a very elementary but very important inequality, called the **arithmetic-geometric inequality**. In these notes this is the first time that we encounter it, but it will come back again and again in the sequel, so we take this opportunity to state it with enough generality. Lemma 5.21 is a corollary of the last part of the following lemma.

Lemma 5.23 (Arithmetic-geometric inequality). *(i) Let* $(x_i)_{1\leq i\leq n}$ *and* $(\lambda_i)_{1\leq i\leq n}$ *be real numbers satisfying*

$$x_i \geq 0, \qquad \lambda_i \geq 0, \qquad \sum_{i=1}^{n} \lambda_i = 1.$$

Then, with the convention $0^0 = 1$,

$$\sum_{i=1}^{n} \lambda_i x_i \geq \prod_{i=1}^{n} x_i^{\lambda_i}.$$

(ii) Let A *and* B *be two nonnegative symmetric* $n \times n$ *matrices (i.e.* $A, B \in S_n^+(\mathbb{R})$*), and* $\lambda \in [0,1]$. *Then*

$$\det\big(\lambda A + (1-\lambda)B\big)^{1/n} \geq \lambda(\det A)^{1/n} + (1-\lambda)(\det B)^{1/n}.$$

(iii) Let $A, B \in S_n^+(\mathbb{R})$ *and* $\lambda \in [0,1]$. *Then*

$$\det\big(\lambda A + (1-\lambda)B\big) \geq (\det A)^{\lambda}(\det B)^{1-\lambda}.$$

Proof. 1. The proof of (i) is an immediate consequence of the concavity of the logarithm function on $\mathbb{R}_+$.

2. To prove (ii), in view of the identity $\det(\lambda A) = \lambda^n(\det A)$ it is sufficient to prove

$$\det(A+B)^{1/n} \geq (\det A)^{1/n} + (\det B)^{1/n}. \tag{5.23}$$

If we can prove this inequality in the particular case when A is invertible, then the general case will follow by density. So let us assume that A is invertible; in view of the identity $\det(MN) = (\det M)(\det N)$, inequality (5.23) will be a consequence of

$$\det(I_n + C)^{1/n} \geq (\det I_n)^{1/n} + (\det C)^{1/n}, \qquad C = A^{-1/2}BA^{-1/2}. \tag{5.24}$$

Note that C is symmetric nonnegative. So the problem is to prove (5.24) for an arbitrary $C \in S_n^+(\mathbb{R})$. For this we can diagonalize C, and introduce its eigenvalues $c_1, \ldots, c_n$, which are nonnegative. Then (5.24) reduces to

$$\prod (1+c_i)^{1/n} \geq 1 + \Big(\prod c_i\Big)^{1/n}.$$

But this is an immediate consequence of (i), since

$$\prod\left(\frac{1}{1+c_i}\right)^{1/n} + \prod\left(\frac{c_i}{1+c_i}\right)^{1/n} \leq \frac{1}{n}\sum \frac{1}{1+c_i} + \frac{1}{n}\sum\frac{c_i}{1+c_i} = 1.$$

3. Finally, (iii) follows from (ii) after raising both sides of the inequality

$$\lambda(\det A)^{1/n} + (1-\lambda)(\det B)^{1/n} \geq (\det A)^{\frac{\lambda}{n}}(\det B)^{\frac{1-\lambda}{n}}$$

to the power n. □

Remark 5.24. In case we consider a strictly concave cost function c instead of the quadratic one, the Jacobian matrix $\nabla\theta$ is not symmetric. However, by Remark 2.56, it is diagonalizable with all of its eigenvalues between 0 and 1, at least when $c \in C^{1,1}_{\text{loc}}(\mathbb{R}^n)$; so the proof applies also in this case.

5.2.4. Potential energy. We now consider part (ii) of Theorem 5.15. Please note that properties of displacement convexity of $\mathcal{V}$ have nothing to do with the fact that $\mathcal{V}$ is formally a linear functional of ρ.

Proof of part (ii) of Theorem 5.15. With the same notation as in the previous subsection,

$$\mathcal{V}(\rho_t) = \int_{\mathbb{R}^n} V\, d[(\mathrm{Id} - t\theta)\#\mu] = \int_{\mathbb{R}^n} V(x - t\theta(x))\, d\mu(x).$$

The convexity of this integral, as a function of t, is an immediate consequence of the convexity of V. Moreover, if V is strictly convex, the convexity of $t \mapsto \mathcal{V}(\rho_t)$ can be degenerate only if $\theta(x) = 0$ for $d\mu$-almost all x, which means of course that $\rho_0 = \rho_1$. Also, if V is λ-uniformly convex, then for all t_1, t_2, σ in $[0,1]$,

$$\sigma\mathcal{V}(\rho_{t_1}) + (1-\sigma)\mathcal{V}(\rho_{t_2}) - \mathcal{V}(\rho_{\sigma t_1 + (1-\sigma)t_2})$$

$$= \int_{\mathbb{R}^n} \Big[\sigma V(x - t_1\theta(x)) + (1-\sigma)V(x - t_2\theta(x)) - V\Big(\sigma(x - t_1\theta(x)) + (1-\sigma)(x - t_2\theta(x))\Big)\Big]\, d\mu(x)$$

$$\geq \lambda\,\frac{\sigma(1-\sigma)}{2}\int_{\mathbb{R}^n} \big|(x - t_1\theta(x)) - (x - t_2\theta(x))\big|^2\, d\mu(x)$$
$$= \lambda\,\frac{\sigma(1-\sigma)}{2}\left[\int_{\mathbb{R}^n} |\theta(x)|^2\, d\mu(x)\right](t_1 - t_2)^2.$$

Since $\int |\theta|^2\, d\mu = \mathcal{T}_2(\mu,\nu)$, we see that $t \mapsto \mathcal{V}(\rho_t)$ is uniformly displacement convex with constant λ. The other cases are proven in the same manner. □

Exercise 5.25 (Necessary condition for displacement convexity of $\mathcal{V}$). Show that, as announced in Theorem 5.15 (ii), the displacement convexity of $\mathcal{V}$ on $P_2(\mathbb{R}^n)$ is in fact equivalent to the convexity of V.

Hint: Interpolate between Dirac masses.

5.2.5. Interaction energy. Finally, consider an interaction potential W, and

$$\mathcal{W}(\rho) = \frac{1}{2}\int_{\mathbb{R}^n\times\mathbb{R}^n} W(x-y)\,d\rho(x)\,d\rho(y).$$

Note that W may be replaced by its symmetric part,

$$W^S(z) = \frac{1}{2}\big[W(z) + W(-z)\big],$$

without affecting the functional $\mathcal{W}$. So we may assume that W is even, $W(-z) = W(z)$.

Important Remarks 5.26. (i) The functional $\mathcal{W}$ is in general *not* convex in the usual sense, *except* for certain particular cases, namely when W is of positive type, i.e. when its Fourier transform is nonnegative. This is the case for instance when W is an *inverse power law*, or a power law under certain restrictions. An example arising in physics is the case $W(z) = |z|^3$, when one restricts $\mathcal{W}$ to probability densities with a fixed mean, in dimension 1 of space. Hence, the plain convexity of $\mathcal{W}$ is extremely sensitive to the particular form of the interaction potential, while the displacement convexity is a much more robust property. On the other hand, the extremely important case of Coulomb interaction, $W(z) = |z|^{-(n-2)}$ ($W(z) = -\log|z|$ in dimension $n = 2$) leads to a functional $\mathcal{W}$ which is convex in the usual sense, but not displacement convex.

(ii) The functional $\mathcal{W}$ is invariant under translation. Namely, if $\tau_a : x \longmapsto x + a$ for some $a \in \mathbb{R}^n$, then $\mathcal{W}(\tau_a\#\rho) = \mathcal{W}(\rho)$. This is why strict displacement convexity can hold only if we rule out translations, for instance by fixing the center of mass of ρ.

Proof of part (iii) of Theorem 5.15. By definition of push-forward,

$$\mathcal{W}(\rho_t) = \frac{1}{2}\int_{\mathbb{R}^n\times\mathbb{R}^n} W\Big([x-y] - t[\theta(x) - \theta(y)]\Big)\,d\mu(x)\,d\mu(y).$$

If W is convex, it follows immediately that $t \longmapsto \mathcal{W}(\rho_t)$ also is. Moreover, if W is strictly convex, then equality in the convexity inequality for $t \mapsto \mathcal{W}(\rho_t)$ occurs if and only if $\theta(x) = a$ for $d\mu$-almost all $x \in \mathbb{R}^n$, where a is some fixed element in $\mathbb{R}^n$. This condition means that μ and ν are *translates of each other*:

$$d\nu(x) = d\mu(x+a).$$

Now, if W is uniformly convex with constant λ, then one easily obtains, in the same manner as in the previous section,

$$\begin{aligned}\sigma\mathcal{W}(\rho_{t_1}) + (1-\sigma)\mathcal{W}(\rho_{t_2}) - \mathcal{W}(\rho_{\sigma t_1+(1-\sigma)t_2}) \\ \geq \frac{1}{2}\lambda\,\frac{\sigma(1-\sigma)}{2}\left[\int_{\mathbb{R}^n\times\mathbb{R}^n}|\theta(x)-\theta(y)|^2\,d\mu(x)\,d\mu(y)\right](t_1-t_2)^2.\end{aligned}$$

We now claim that the condition of common center of mass,

$$\int_{\mathbb{R}^n} x\,d\mu(x) = \int_{\mathbb{R}^n} y\,d\nu(y), \tag{5.25}$$

implies

$$\int_{\mathbb{R}^n\times\mathbb{R}^n}|\theta(x)-\theta(y)|^2\,d\mu(x)\,d\mu(y) = 2\int_{\mathbb{R}^n}|\theta|^2\,d\mu = 2\,\mathcal{T}_2(\mu,\nu). \tag{5.26}$$

Indeed, since $(\mathrm{Id}-\theta)\#\mu=\nu$, we have $\int(x-\theta)\,d\mu = \int y\,d\nu$, and from (5.25) we deduce

$$\int \theta\,d\mu = 0.$$

Then, by expanding the square, we find that

$$\begin{aligned}\int_{\mathbb{R}^n\times\mathbb{R}^n}|\theta(x)-\theta(y)|^2\,d\mu(x)\,d\mu(y) &= 2\int_{\mathbb{R}^n}|\theta|^2\,d\mu - 2\left|\int_{\mathbb{R}^n}\theta\,d\mu\right|^2 \\ &= 2\int_{\mathbb{R}^n}|\theta|^2\,d\mu.\end{aligned}$$

This concludes the proof. □

Remark 5.27. By Jensen's inequality, for all convex functions c,

$$\int\theta(y)\,d\mu(y)=0 \\ \Longrightarrow \quad \int_{\mathbb{R}^n\times\mathbb{R}^n} c\big(\theta(x)-\theta(y)\big)\,d\mu(x)\,d\mu(y) \geq \int_{\mathbb{R}^n} c(\theta)\,d\mu.$$

But in the particular case when c is the square norm, this inequality can be improved by a factor 2, as we just saw.

Exercise 5.28 (Necessary condition for displacement convexity of $\mathcal{W}$). Assume that W is even ($W(-z)=W(z)$) and continuous on $\mathbb{R}^n$, and that $n\geq 2$. Show that the displacement convexity of $\mathcal{W}$ on $P_2(\mathbb{R}^n)$ implies the convexity of W, as announced in Theorem 5.15 (iii).

Hint: Look at convex combinations of two Dirac masses. Explain why the reasoning fails if $n=1$.

5.2.6. Above-tangent formulation. It is a general fact that if a function $\Phi : [0,1] \to \mathbb{R} \cup \{+\infty\}$ is λ-uniformly convex, or semiconvex with constant $-\lambda$, then

$$(5.27)\qquad \Phi(1) \geq \Phi(0) + \frac{d}{dt}\Big|_{t=0}^{+} \Phi(t) + \frac{\lambda}{2},$$

where

$$\frac{d}{dt}\Big|_{t=0}^{+} \Phi(t) = \limsup_{t\downarrow 0^+} \frac{\Phi(t) - \Phi(0)}{t}$$

is the upper right derivative of φ at $t = 0$. By semi-convexity of Φ, this lim sup is in fact a true limit. Formula (5.27), combined with the definitions of displacement convexity and its variants, leads to the following proposition.

Proposition 5.29 (Above-tangent formulation of displacement convexity). *Let F be a functional with values in $\mathbb{R} \cup \{+\infty\}$, defined on some displacement convex subset $\mathcal{P}$ of either $P_{\mathrm{ac}}(\mathbb{R}^n)$ or $P_2(\mathbb{R}^n)$. Assume that F is λ-uniformly displacement convex, for some $\lambda > 0$; or semi-displacement-convex with constant $-\lambda \geq 0$ (in short: there exists $\lambda \in \mathbb{R}$ such that $(d^2/dt^2)F(\rho_t) \geq \lambda W_2(\rho_0,\rho_1)^2$ for all ρ_0, ρ_1). Let ρ_0 and ρ_1 be in $\mathcal{P}$ and let $(\rho_t)_{0\leq t\leq 1}$ be their displacement interpolation. Then,*

$$F(\rho_1) \geq F(\rho_0) + \frac{d}{dt}\Big|_{t=0}^{+} F(\rho_t) + \frac{\lambda}{2}\,\mathcal{T}_2(\rho_0,\rho_1).$$

To apply Proposition 5.29 it is desirable to have an explicit expression of $\frac{d}{dt}\big|_{t=0}^{+} F(\rho_t)$. This is what the following theorem provides.

Theorem 5.30 (Practical computation of the tangent). *Let $U : \mathbb{R}_+ \to \mathbb{R} \cup \{+\infty\}$, $V, W : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be measurable functions such that U satisfies the convexity condition (5.17), V and W are convex, and W is symmetric. Let ρ_0, ρ_1 be two absolutely continuous probability measures on $\mathbb{R}^n$, such that $U(\rho_0)$, $U(\rho_1)$, $\rho_0 V$, $\rho_1 V$ belong to $L^1(\mathbb{R}^n)$ and $\rho_0(x)\rho_0(y)W(x-y)$, $\rho_1(x)\rho_1(y)W(x-y)$ belong to $L^1(\mathbb{R}^n \times \mathbb{R}^n, dx\,dy)$. Let $\nabla\varphi$ be the unique gradient of a convex function which pushes ρ_0 forward to ρ_1. Then,*

$$(5.28)\qquad \frac{d}{dt}\Big|_{t=0} \mathcal{U}(\rho_t) = \int_{\mathbb{R}^n} [U(\rho_0) - \rho_0 U'(\rho_0)]\,(\Delta_A\varphi - n) = -\int_{\mathbb{R}^n} P(\rho_0)\,(\Delta_A\varphi - n),$$

where Δ_A stands for the Laplace operator in the Aleksandrov sense;

$$(5.29)\qquad \frac{d}{dt}\Big|_{t=0} \mathcal{V}(\rho_t) = \int_{\mathbb{R}^n} \rho_0(x)\,\nabla V(x)\cdot(\nabla\varphi(x) - x)\,dx;$$

and

$$(5.30)\quad \left.\frac{d}{dt}\right|_{t=0} \mathcal{W}(\rho_t) = \int_{\mathbb{R}^n\times\mathbb{R}^n} \rho_0(x)\rho_0(y)\,\nabla W(x-y)\cdot\big[(\nabla\varphi(x)-x)-(\nabla\varphi(y)-y)\big]\,dx\,dy.$$

If, on the other hand, V and W are not convex, but only semiconvex, and $\mathcal{T}_2(\rho_0,\rho_1)<+\infty$, then equations (5.29), (5.30) still hold true with the derivative replaced by an upper derivative, and the equality sign replaced by "$\geq$". For instance,

$$(5.31)\qquad \left.\frac{d}{dt}\right|_{t=0}^{+} \mathcal{V}(\rho_t) \geq \int_{\mathbb{R}^n} \rho_0(x)\,\nabla V(x)\cdot(\nabla\varphi(x)-x)\,dx.$$

Remark 5.31. Without the various convexity assumptions on U, V, W, the conclusions of the theorem may still hold true in the presence of enough regularity for ρ_0, ρ_1 and/or φ. This should be checked on each case of interest.

Proof of Theorem 5.30. We start with (5.28). Using formula (4.11), we rewrite

$$(5.32)\quad \frac{\mathcal{U}(\rho_t)-\mathcal{U}(\rho_0)}{t} = \int_{\mathbb{R}^n} \frac{1}{t}\left\{U\left(\frac{\rho_0(x)}{\det\big[(1-t)I_n+tD_A^2\varphi(x)\big]}\right) \times \det\big[(1-t)I_n+tD_A^2\varphi(x)\big] - U(\rho_0(x))\right\}dx,$$

which we rewrite (with obvious notation)

$$\int_{\mathbb{R}^n} \frac{1}{t}\Big\{u(t,x)-u(0,x)\Big\}\,dx.$$

From our assumptions, both $u(1,\cdot)$ and $u(0,\cdot)$ are integrable. For almost all x, the function $t\longmapsto u(t,x)$ is well-defined and convex; therefore its slope $(u(t,x)-u(0,x))/t$ is nonincreasing as $t\downarrow 0^+$, and converges monotonically to $u'(0,x)$, where $'$ stands for the partial derivative with respect to t. It is a good exercise to check that

$$u'(0,x) = \big[U(\rho_0(x))-\rho_0(x)U'(\rho_0(x))\big](\Delta_A\varphi(x)-n).$$

Then the conclusion follows by an application of the monotone convergence theorem.

We now turn to (5.29): for this we use the definition of push-forward to write

$$\frac{\mathcal{V}(\rho_t)-\mathcal{V}(\rho_0)}{t}=\int \rho_0(x)\left[\frac{V\big((1-t)x+t\nabla\varphi(x)\big)-V(x)}{t}\right].$$

If V is convex, then the term in square brackets converges monotonically to $\nabla V(x)\cdot\nabla\varphi(x)$, for almost all x, and the conclusion follows by monotone convergence again. On the other hand, if V is semiconvex, then there exists a constant $C\in\mathbb{R}$ such that, say for $t<1/2$,

$$\frac{\mathcal{V}(\rho_t)-\mathcal{V}(\rho_0)}{t}\geq \int \rho_0(x)\,\nabla V(x)\cdot(\nabla\varphi(x)-x)\,dx-C\int \rho_0(x)|\nabla\varphi(x)-x|^2\,dx. \tag{5.33}$$

The assumption $\mathcal{T}_2(\rho_0,\rho_1)<+\infty$ implies that the function $x\mapsto|\nabla\varphi(x)-x|^2$ lies in $L^1(\rho_0(x)\,dx)$; therefore, Fatou's lemma can be applied to the left-hand side of (5.33), and we arrive at (5.31).

The proof of (5.30) follows the same lines, and is left as an exercise. □

5.2.7. Riemannian manifolds. Most, if not all, of the preceding discussion still makes sense on a Riemannian manifold M. But the results may be deeply modified! In a genuinely Riemannian setting, the ambient geometry may come into play in the properties of displacement convexity; and in particular the **Ricci curvature** plays a crucial role. For instance, one can show that an internal energy satisfying the convexity condition (5.17) is still displacement convex on a manifold of nonnegative Ricci curvature; or that a potential energy with potential V is displacement convex if the sum $D^2V+\mathrm{Ric}$ is nonnegative, where D^2V stands for the Hessian of V and Ric for the Ricci tensor of the manifold.

It is out of the scope of these notes to define and to discuss the properties of the Ricci curvature; for this we refer to a treatise in differential geometry, such as [**93**], [**121**] or [**147**]. As we shall see in the last section of this chapter, it is possible to have a purely analytical approach to these results. The first conjectures about displacement convexity on manifolds appeared in [**219**], and the first rigorous foray into the field was due to Cordero-Erausquin, McCann and Schmuckenschläger [**96, 100**]. Since then the field has grown to incredible extents; an enormous part of [**267**] is devoted to this rich topic, and a number of courses and synthesis documents have also been written about it.

5.3. Application: uniqueness of ground state

A basic but important application of Theorem 5.15, which actually was the initial motivation for this theorem, is the following uniqueness result by McCann [**203**].

Theorem 5.32 (Strict displacement convexity implies uniqueness of minimizer). *Consider the following energy functional, defined for absolutely continuous probability measures on* $\mathbb{R}^n$*:*

$$(5.34)\quad F(\rho) = \int_{\mathbb{R}^n} U\big(\rho(x)\big)\, dx + \int_{\mathbb{R}^n} V\, d\rho + \frac{1}{2}\int_{\mathbb{R}^n\times\mathbb{R}^n} W(x-y)\, d\rho(x)\, d\rho(y).$$

Assume that U *satisfies condition* (5.17), $\inf V > -\infty$, *and* V *and* W *are convex. Assume moreover that* V *(resp.* W*) is strictly convex. Then, there is at most one minimizer (resp. at most one minimizer, possibly up to translation) for* (5.34) *on the set of absolutely continuous probability measures on* $\mathbb{R}^n$.

Proof. The proof is immediate with the help of the machinery developed above. Let ρ_1, ρ_2 be two distinct absolutely continuous minimizers, and consider $\rho = [\rho_1, \rho_2]_{1/2}$. By Theorem 5.15, F is strictly displacement convex; hence $t \mapsto F([\rho_1, \rho_2]_t)$ is strictly convex, with the possible exception of the case when V is not strictly convex and ρ_1, ρ_2 are translates of each other. But strict convexity implies $F(\rho) < [F(\rho_1) + F(\rho_2)]/2$, which is impossible since ρ_1 and ρ_2 are minimizers. □

We conclude this section with a long series of remarks, all of which were already formulated by McCann:

Remarks 5.33. (i) This scheme of proof is standard for convex (in the usual sense) functionals. Here the notion of displacement convexity enables one to bypass the lack of convexity.

(ii) As a corollary of the uniqueness result, we see that the minimizer has to be even (possibly up to translation) if V is even. Recall that, without loss of generality, W may be assumed to be even. Similarly, if V and W are radially symmetric, then the minimizer has to be radially symmetric, possibly up to translation.

(iii) We did not use the Euler-Lagrange equation associated to the minimization problem (see [**133**] for an elementary introduction to the theory of Euler-Lagrange equations). In the absence of the structure of displacement convexity, a natural strategy for trying to prove uniqueness would be to go through the study of the uniqueness of this Euler-Lagrange equation, which is usually not a simple thing. Another possible strategy, if W is

radially symmetric, would be to search for a **strict rearrangement inequality**, i.e. prove that $\mathcal{W}(\rho^*) < \mathcal{W}(\rho)$ if ρ is not monotone radially symmetric. Here ρ^* is the monotone radially symmetric rearrangement of ρ; note that $\mathcal{U}(\rho^*) = \mathcal{U}(\rho)$ automatically. For instance, the strict Riesz rearrangement inequality of [**190**, Theorem 3.9] implies the following: if W is radially symmetric and $W(z)$ is strictly decreasing as a function of $|z|$, then $\mathcal{W}(\rho^*) < \mathcal{W}(\rho)$ unless $\rho = \rho^*(\cdot - a)$ for some $a \in \mathbb{R}^n$. (Of course, this theorem does not apply in the present situation since W is not decreasing, but it can serve as an inspiration...) After that, the problem would be reduced to a one-dimensional problem, which one can try to solve by ordinary differential equation methods, see [**191**]. Note however that even if we managed to fulfill this program, the solution of the one-dimensional problem would in fact be much simpler via displacement convexity.

(iv) What about the *existence* of an absolute minimizer? The energy functional (5.34) is not convex in the usual sense, but it is lower semi-continuous with respect to the weak-$*$ topology, because weak convergence of a sequence (ρ_k) implies weak convergence of $(\rho_k \otimes \rho_k)$ in $P(\mathbb{R}^n \times \mathbb{R}^n)$ (for instance, this can be seen as a consequence of the density of the vector space generated by tensor products $\varphi \otimes \psi$ in $C_0(X \times Y)$). We can always assume that F is bounded below, because of the condition $\inf V > -\infty$ and because W can be chosen even, in which case it takes its minimum at the origin. Assume also for simplicity that $V(z) \to +\infty$ as $|z| \to \infty$. Then (ρ_k) is tight, and up to extraction of a subsequence we know that ρ_k converges weakly to some probability measure ρ_∞. But we need to ensure that ρ_∞ is absolutely continuous! A sufficient condition (definitely not necessary) is that

$$\lim_{r \to +\infty} \frac{U(r)}{r} = +\infty. \tag{5.35}$$

At the level of the pressure, a sufficient condition for (5.35) is that $P(\rho)/\rho^2$ be nonintegrable as $\rho \to \infty$. Note, however, that condition (5.35) excludes part of the choice which was available in the examples following (5.20).

(v) In the case when V and W are not strictly convex, one may use a property of strict displacement convexity of the internal energy $\mathcal{U}$. For instance, McCann [**203**] shows that if $P(\rho)/\rho^{1-1/n}$ is *strictly* increasing from 0, and W is convex, radially symmetric, nonconstant, then again there is at most one minimizer, up to translation, for the functional $\mathcal{U} + \mathcal{W}$.

A related study of the equilibrium shape of crystals was performed by McCann in [**204**]. In that paper the reader can find a lot of background on minimization problems arising in this area, and their links with isoperimetric problems.

5.4. The Eulerian point of view

In the beginning of this chapter, we have discussed the time-dependent minimization problem from a *Lagrangian* point of view, i.e. taking into account the collection of all trajectories. Now we would like to have a picture of the velocity field of the particles in this interpolation process: an *Eulerian* point of view. We start with a rather informal discussion, and give rigorous justification only at the end of the section.

5.4.1. Eulerian/Lagrangian duality. To switch between Eulerian and Lagrangian formulations, one just has to set

$$v(t, g(t,x)) = \frac{dg}{dt}(t,x),$$

where v is the time-dependent velocity field and $g(t,x)$ the family of trajectories. This makes sense for instance if v is of class C^1, or at least locally Lipschitz in x (so that the Cauchy-Lipschitz theory applies), and $(g(t,\cdot))_{t\geq 0}$ is a family of diffeomorphisms.

Here, we are interested in $g(t,x) = T_t(x)$: particles follow optimal paths. But it is not a priori clear that these paths define a family of diffeomorphisms. In the case when the cost is quadratic and when Caffarelli's regularity theory for the Monge-Ampère equation applies (recall subsection 4.2.2 in Chapter 4), this is obviously true under suitable assumptions on μ and ν. As we shall see, even if we cannot guarantee that (T_t) are C^1 diffeomorphisms, we can show that they are injective in two situations of interest: for homogeneous costs in $\mathbb{R}^n$, and for the quadratic cost on a manifold; and we shall also give a proof that they are locally Lipschitz for $0 < t < 1$.

One would a priori expect considerable regularity problems here. Let us consider for instance the case when Theorem 5.5 applies. Then, the velocity field at time 0 coincides with $-\nabla c^*(\nabla\psi)$. This will not be smooth in general; given that ψ is semiconcave, one can at best hope for a one-sided regularity condition of the type

$$\Big\langle \nabla c^*(\nabla\psi(x)) - \nabla c^*(\nabla\psi(y)),\ x-y\Big\rangle \leq C\,|x-y|^2 \tag{5.36}$$

(see Section 2.5 in Chapter 2). But this is insufficient to apply the classical theory of characteristics and to switch from Eulerian to Lagrangian formulations – even if refinements of the theory, which cover such assumptions, have been studied by Filippov [**141, 142**], see also [**55**]. However, a somewhat surprising feature of the problem is that the velocity field will be better behaved, for times $t \in (0,1)$, than it is a priori at initial and final time: some regularization process is going on.

To caricature things a little bit, the Lagrangian formulation is "stronger" than the Eulerian one, but usually breaks down because of the problem of

shocks, i.e. the meeting of different trajectories. For most equations of interest in compressible fluid mechanics, shocks appear in finite time, even for very smooth initial data. However, this is not the case in the problem of optimal transportation, at least in the important case of a quadratic cost function. In fact, unlike many other related problems, *the mass transportation problem is best solved with a Lagrangian point of view.* So our justification of the Eulerian point of view will be entirely based on the Lagrangian one.

Why bother with an Eulerian representation, if it performs less well and cannot be justified rigorously without the help of the Lagrangian machinery? It will turn out that the discussion of the Eulerian point of view is incredibly rich, and furnishes remarkable help for the intuition, as well as a link to other problems involving partial differential equations. Therefore it will be a rewarding investment to investigate it.

5.4.2. Optimality equations. Let $(T_t)_{0\leq t\leq 1}$ be the solution of the time-dependent minimization problem (5.2), and consider the probability measure "at intermediate times",

$$\rho_t = T_t\#\mu.$$

What is the natural evolution equation for (ρ_t)? It can be expressed in terms of the velocity field itself. This is the content of the following theorem, which lies at the basis of the theory of linear transport equations. It is a particular case of the theory of **characteristic equations**, which is expounded in [**133**, Section 3.2] for instance.

Theorem 5.34 (Characteristics method for linear transport equations). *Let X be $\mathbb{R}^n$, or more generally a smooth complete manifold. Let $(T_t)_{0\leq t<T_*}$ be a locally Lipschitz family of diffeomorphisms in X, with $T_0 = \mathrm{Id}$, and let $v = v(t,x)$ be the velocity field associated with the trajectories (T_t). Let μ be a probability measure on X, and $\rho_t = T_t\#\mu$. Then, $\rho_t = \rho(t,\cdot)$ is the unique solution of the linear transport equation*

$$\begin{cases} \dfrac{\partial \rho}{\partial t} + \nabla\cdot(\rho v) = 0, \qquad 0 < t < T_*, \\ \\ \rho_0 = \mu \end{cases} \tag{5.37}$$

in $C([0,T_*);P(X))$, where $P(X)$ is equipped with the weak topology.

Remarks 5.35. (i) When we say that (T_t) is a locally Lipschitz family of diffeomorphisms, we mean that (a) T_t it is a bijection $X \to X$ for all t, and (b) for all $T < T_*$ and for all compact $K \subset X$, the maps $(t,x) \longmapsto T_t(x)$ and $T_t^{-1}(x)$ are Lipschitz on $[0,T]\times K$. For instance, if a velocity field

$v = v(t, x)$ is given and uniformly Lipschitz on $\mathbb{R}_+ \times \mathbb{R}^n$, then by Cauchy-Lipschitz theory it generates a family of (globally) Lipschitz diffeomorphisms for all times.

(ii) The **divergence** operator $\nabla\cdot$ is defined by duality, through the formula

$$\int \varphi\, d(\nabla \cdot m) = -\int \nabla\varphi \cdot dm,$$

where φ is a smooth test function with compact support and m a vector-valued measure. Of course, if v is smooth and the setting is Euclidean, then

$$\nabla \cdot v = \sum_{i=1}^{n} \frac{\partial v_i}{\partial x_i}.$$

(iii) Equation (5.37) is known in physics as the identity of **conservation of mass**, and motivates our notation ρ (reminiscent of the standard notation for the density of a fluid). Compare with our discussion of the incompressibility condition in Section 3.2 of Chapter 3.

(iv) In the case when X has boundaries, then equation (5.37) should be complemented by boundary conditions. As far as mass transportation is concerned, some subtle problems arise here, see the warning in subsection 5.4.3 below.

Proof of Theorem 5.34. The proof is rather instructive, and it is useful to know at least the general principle of it. We write $v_t(x) = v(t, x)$.

1. First, let us show that $\rho_t = T_t\#\mu$ does solve (5.37). We shall show that for all $\varphi \in \mathcal{D}(X)$ and $T \in (0, T_*)$, the mapping $t \longmapsto \int \varphi\, d\rho_t$ is Lipschitz on $(0, T)$, with derivative

$$\frac{d}{dt}\int \varphi\, d\rho_t = -\int \varphi\, d[\nabla \cdot (v_t\rho_t)] \equiv \int (\nabla\varphi \cdot v_t)\, d\rho_t$$

for almost all t. For this we first use the definition of push-forward to write

$$\int \varphi\, d\rho_t = \int (\varphi \circ T_t)\, d\mu.$$

Since φ is compactly supported and T_t^{-1} is continuous, the function $\varphi \circ T_t$ is supported in a compact set (uniform for $0 \le t \le T$); moreover it is Lipschitz and, for almost all t, x,

$$\frac{\partial}{\partial t}(\varphi \circ T_t) = (\nabla\varphi \circ T_t) \cdot \frac{\partial T_t}{\partial t} = (\nabla\varphi \circ T_t) \cdot (v_t \circ T_t).$$

Then, for $h > 0$ we can write

$$\frac{1}{h}\left(\int \varphi\, d\rho_{t+h} - \int \varphi\, d\rho_t\right) = \int \left(\frac{\varphi \circ T_{t+h}(x) - \varphi \circ T_t(x)}{h}\right) d\mu.$$

The integrand inside the parentheses in the right-hand side is uniformly bounded on $[0, T-h] \times \mathbb{R}^n$, and for almost all t it converges to $(\nabla\varphi \circ T_t) \cdot v_t$ as $h \to 0$, for almost all x. By Lebesgue's dominated convergence theorem, we deduce that the map $t \longmapsto \int \varphi \, d\rho_t$ is differentiable for almost all t, and

$$\begin{aligned} \frac{d}{dt} \int \varphi \, d\rho_t &= \int (\nabla\varphi \circ T_t) \cdot (v_t \circ T_t) \, d\mu \\ &= \int \nabla\varphi \cdot v_t \, d\rho_t, \end{aligned}$$

which is what we were looking for. The continuity of $T_t\#\mu$, as a function of t, with respect to the weak topology of measures, is left as an exercise.

2. Now we turn to uniqueness. By linearity, it is sufficient to prove that if a time-dependent measure (ρ_t) (satisfying the regularity conditions of the theorem) solves (5.37), then for all $T < T_*$,

$$\rho_0 = 0 \implies \rho_T = 0.$$

For that we shall use a **duality method**. Assume that we can construct a Lipschitz function $\varphi(t, x)$, defined on the time-interval $[0, T]$, compactly supported, and solving

$$\begin{cases} \dfrac{\partial\varphi}{\partial t} = -v \cdot \nabla\varphi, \\ \\ \varphi|_{t=T} = \varphi_T, \end{cases} \tag{5.38}$$

where φ_T is an arbitrary function in $\mathcal{D}(X)$. Then, by an argument similar to the one above, we deduce that $t \longmapsto \int \varphi_t \, d\rho_t$ is Lipschitz and satisfies

$$\begin{aligned} \frac{d}{dt} \int \varphi_t \, d\rho_t &= \int \frac{\partial\varphi_t}{\partial t} \, d\rho_t + \int \varphi_t \, d\left(\frac{\partial\rho_t}{\partial t}\right) \\ &= -\int v_t \cdot \nabla\varphi_t \, d\rho_t + \int \varphi_t \, d[\nabla \cdot (v_t\rho_t)] = 0 \end{aligned}$$

for almost all t; so

$$\int \varphi_T \, d\rho_T = \int \varphi_0 \, d\rho_0 = 0$$

(here we used the continuity up to $t = 0$). Since φ_T is arbitrary, this implies $\rho_T = 0$.

It remains to construct a solution to (5.38), which can be rewritten as $\partial\varphi/\partial t + v \cdot \nabla\varphi = 0$, or $(d/dt)\varphi_t(T_t x) = 0$. Such a solution should satisfy

$$\varphi_t(T_t x) = \varphi_T(T_T x),$$

or

$$\varphi_t = \varphi_T \circ T_T \circ T_t^{-1}.$$

Since (T_t) is a locally Lipschitz family, the above formula does define a Lipschitz function with compact support, satisfying (5.38) almost everywhere. □

Remark 5.36. A key parameter in the qualitative behavior of the flow is the *divergence* of the vector field v. If it is 0, the flow is incompressible (as we saw in Chapter 3); if it is negative, then the flow is *contracting* and tends to create higher densities as time goes by; if it is positive, then the flow is *expanding* and tends to create lower densities.

Exercise 5.37 (Transport of density). Assume that $d\rho_t(x) = f_t(x)\,dx$ and use Theorem 5.34 to write down an equation satisfied by f_t. Give a direct proof that f_t does satisfy this equation. Show that in the particular case when $\nabla \cdot v = 0$, i.e. when the flow is **incompressible**, then not only is the probability measure ρ_t transported, but also its density f_t:

$$f_t(T_t x) = f_0(x).$$

This fact is at the basis of many numerical methods for transport equations, like the Vlasov equation in plasma physics.

Hint: Recall equation (15).

To get back to our problem, we now wish to identify the velocity field $v(t,x)$ corresponding to optimal transportation. Let us consider first the case of a strictly convex cost function in a Euclidean space. As we have already seen, in Lagrangian representation the optimal trajectories are just straight lines (Theorem 5.5).

Proposition 5.38 (Eulerian representation for geodesic trajectories). *Let $v_0 : \mathbb{R}^n \to \mathbb{R}^n$ be a continuous function on $\mathbb{R}^n$, differentiable almost everywhere, and let $T_t(x) = x - tv_0(x)$ be a field of trajectories of particles, each of them moving with constant velocity. Assume that $(T_t)_{0\le t<T_*}$ defines a family of diffeomorphisms. Then for $0 \le t < T_*$ the associated Eulerian velocity field $v_t = dT_t/dt \circ T_t^{-1}$ satisfies the equation*

$$\frac{\partial v}{\partial t} + v \cdot \nabla v = 0. \tag{5.39}$$

Remark 5.39. Compare equation (5.39) with (3.4). Here there is no incompressibility constraint: particles ignore each other, in some sense, and as a consequence there is no Lagrange multiplier (no pressure!). Consequently, equation (5.39) is called the **pressureless Euler equation**. In one dimension of space, it is well-known under the name of **Burgers' equation**. Even in this one-dimensional case, it is trickier than it seems, because of the problem of shocks (see below).

Proof of Proposition 5.38. By differentiating $(d/dt)T_t(x) = v(t, T_t(x))$, we obtain, for any given x,

$$0 = \frac{d^2}{dt^2}(T_t x) = \frac{\partial v}{\partial t}(T_t x) + v(t, T_t x) \cdot \nabla v(t, T_t x).$$

□

Combining (5.37) and (5.39), we obtain the **Eulerian system of optimal time-dependent mass transportation**:

$$\begin{cases} \dfrac{\partial \rho}{\partial t} + \nabla \cdot (\rho v) = 0, \qquad \rho(t = 0, \cdot) = \mu, \\ \\ \dfrac{\partial v}{\partial t} + v \cdot \nabla v = 0. \end{cases} \tag{5.40}$$

This system is formally equivalent to

$$\begin{cases} \dfrac{\partial \rho}{\partial t} + \nabla \cdot (\rho v) = 0, \qquad \rho(t = 0, \cdot) = \mu, \\ \\ \dfrac{\partial (\rho v)}{\partial t} + \nabla \cdot (\rho v \otimes v) = 0, \end{cases} \tag{5.41}$$

which is called the **sticky particles system**, and describes the usual compressible Euler equations at *zero temperature*.

Remarks 5.40. (i) System (5.41) is very tricky, and exhibits some kinds of instability phenomena. A proof of uniqueness of its solutions in one dimension of space was obtained by Bouchut and James [**55**], while existence results go back to Grenier [**158**] (see also E, Rykov and Sinai [**128**]). A simpler proof of existence in dimension one was given later in Brenier and Grenier [**67**], while a paper by Sever [**237**] is concerned with existence theorems in a multidimensional setting. The reason why we do not see all the analytical problems which were encountered by these authors, is that we consider a very peculiar situation, in which shocks do not arise (see below), and the velocity field has a peculiar structure.

(ii) The denomination of "sticky particles" cannot be understood if one does not care about shocks! This terminology comes from the dynamics of the system when particles encounter each other: they stick together and form a bigger particle, whose velocity is determined by the law of conservation of momentum.

(iii) It is important to note that solutions of system (5.41) preserve the "kinetic energy":

$$\int_{\mathbb{R}^n} |v_t|^2 \, d\rho_t(x) = \int_{\mathbb{R}^n} |v_0|^2 \, d\rho_0(x). \tag{5.42}$$

Again, this is a consequence of the absence of shocks.

(iv) It may seem surprising that in (5.40), *we have lost the transportation cost*! This set of equations is valid whatever the strictly convex cost c. Actually the cost function is hidden in the initial conditions for v. This will be apparent from the next proposition, which for simplicity is stated only for smooth data.

Proposition 5.41 (Optimal initial velocity field). *Assume that we are given a smooth solution of the Eulerian system* (5.40). *Then, the associated Lagrangian field of trajectories determines an optimal transportation for the cost c if and only if*

$$v(t=0,\cdot) = -\nabla c^*(\nabla\psi) \tag{5.43}$$

for some c-concave function ψ.

The proof is just a rewriting of the considerations above, and is a good exercise for the reader.

Example 5.42. In the case $c(z) = |z|^2$, then $v(t=0,\cdot)$ should be $\nabla\varphi - \mathrm{Id}$, where $\nabla\varphi$ is the gradient of a convex function which pushes $\rho(t=0,\cdot)$ forward to $\rho(t=1,\cdot)$.

A further reduction can be performed, starting from (5.39). Let us look for a function $u = u(t,x)$ such that

$$v = \nabla c^*(\nabla u). \tag{5.44}$$

It is an easy exercise in differential calculus to check that v satisfies (5.39) if u satisfies the **Hamilton-Jacobi equation** with Hamiltonian function c^*,

$$\frac{\partial u}{\partial t} + c^*(\nabla u) = 0. \tag{5.45}$$

With these new unknowns, the optimality equations read

$$\begin{cases} \dfrac{\partial \rho}{\partial t} + \nabla\cdot\big(\rho\nabla c^*(\nabla u)\big) = 0, \qquad \rho(t=0,\cdot) = \mu, \\[2ex] \dfrac{\partial u}{\partial t} + c^*(\nabla u) = 0. \end{cases} \tag{5.46}$$

Moreover, if ∇c^* is an odd function, then the optimal condition becomes

$$-u(t=0,\cdot) \ \text{ is } c\text{-concave},$$

as shown by (5.43) and (5.44).

5.4.3. Boundary conditions. Important difficulties may arise concerning boundary conditions — even in the simple case of a bounded smooth subset Ω of $\mathbb{R}^n$, say a ball. The reason is that, even if μ and $\nu = T_1\#\mu$ are strictly positive in Ω, as a general rule, the interpolant ρ_t at time $t \in (0,1)$ will have a strictly smaller support than ρ_0 and ρ_1. In fact, this happens as soon as the map T_1 does not leave the boundary $\partial\Omega$ pointwise invariant (exercise), which is the general case. The resulting equations are rather intricate.

5.4.4. Distance cost function and linear interpolation. Consider now the case when $c(x,y) = |x-y|$ in $\mathbb{R}^n$, and remember from subsection 2.4.6 that the optimal transportation can be obtained by solving the equation

$$-\nabla \cdot (a\nabla u) = \mu - \nu. \tag{5.47}$$

As used in [**135**], the corresponding Eulerian formulation turns out to coincide with the linear interpolation,

$$\rho_t = (1-t)\mu + t\nu.$$

Indeed, let f and g stand for the respective densities of μ and ν, and define v by

$$v = \frac{a\nabla u}{(1-t)f + tg}.$$

Then, from (5.47) we see that

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho v) = 0.$$

Thus the linear interpolation appears as a particular case of Monge-Kantorovich interpolation. From a Lagrangian point of view, this is an extremely degenerate situation. For instance, if μ and ν have disjoint supports, then the mass is transferred "instantaneously" from μ to ν.

5.4.5. Displacement convexity from the Eulerian point of view. For several applications to partial differential equations in divergence form, it is important to master the Eulerian point of view in the study of displacement convexity, even at a formal level. To perform explicit calculations, the simplest way may be to use the optimality equations in the form

$$\begin{cases} \dfrac{\partial \rho}{\partial t} + \nabla \cdot (\rho v) = 0, \\[2ex] \dfrac{\partial (\rho v)}{\partial t} + \nabla \cdot (\rho v \otimes v) = 0, \end{cases} \tag{5.48}$$

with a velocity field $v = \nabla c^*(\nabla u)$. In the sequel, we again assume that we deal with "smooth" solutions, and we use the notation $\rho_t(x) = \rho(t,x)$, $v_t(x) = v(t,x)$.

We first consider the most complicated case, namely the internal energy $\mathcal{U}(\rho)$. After a long calculation, one finds that

$$(5.49) \qquad \frac{d}{dt}\int U\big(\rho_t(x)\big)\,dx = -\int U'\big(\rho_t(x)\big)\,\nabla\cdot(\rho_t v_t)\,dx,$$

$$(5.50) \qquad \frac{d^2}{dt^2}\int U\big(\rho_t(x)\big)\,dx = \int \big[\rho_t P'(\rho_t) - P(\rho_t)\big](\mathrm{tr}\ \nabla v_t)^2\,dx + \int P(\rho_t)\,\mathrm{tr}\,[(\nabla v_t)^2]\,dx.$$

Here, as before, ∇v stands for the Jacobian matrix of v,

$$(\nabla v)_{ij} = \frac{\partial v_i}{\partial x_j}.$$

What replaces Lemma 5.21 in this context is now

Lemma 5.43 (Concavity of $\det^{1/n}$, differential version). *Let A be a diagonalizable $n\times n$ matrix. Then*

$$\mathrm{tr}\,(A^2) \ge \frac{1}{n}(\mathrm{tr}\ A)^2.$$

Proof of Lemma 5.43. This is just a simple consequence of the Cauchy-Schwarz inequality. □

Recall from Remark 2.56 that ∇v is diagonalizable, so that Lemma 5.43 can be used in our case. This and the positivity of the pressure P imply

$$(5.51) \qquad \frac{d^2}{dt^2}\int U(\rho_t(x))\,dx \ge \int \left[\rho_t P'(\rho_t) - \left(1-\frac{1}{n}\right)P(\rho_t)\right](\mathrm{tr}\ \nabla v_t)^2\,dx.$$

Thus we find again that $\mathcal{U}$ is displacement convex when the the density of internal energy U fulfills condition (5.20).

Remark 5.44. In the important case $U(\rho) = \rho\log\rho$, one of the terms in (5.50) vanishes and we are simply left with

$$(5.52) \qquad \frac{d^2}{dt^2}\int \frac{d\rho_t}{dx}\log\frac{d\rho_t}{dx}\,dx = \int \mathrm{tr}\,(\nabla v)^2\,d\rho_t.$$

The treatment of the potential energy $\mathcal{V}$ and of the interaction energy $\mathcal{W}$ from an Eulerian viewpoint is simpler: after some calculations, one finds that

$$\frac{d}{dt}\mathcal{V}(\rho_t) = \int (v_t\cdot\nabla V)\,d\rho_t,$$

$$\frac{d^2}{dt^2}\mathcal{V}(\rho_t) = \int \langle D^2V\cdot v_t, v_t\rangle\,d\rho_t,$$

and

$$\frac{d}{dt}\mathcal{W}(\rho_t) = \int \left[v_t \cdot \nabla(W * \rho_t)\right] d\rho_t,$$

$$\frac{d^2}{dt^2}\mathcal{W}(\rho_t) = \frac{1}{2}\int \Big\langle D^2W(x-y)\cdot[v_t(x)-v_t(y)],\, [v_t(x)-v_t(y)]\Big\rangle\, d\rho_t(x)\, d\rho_t(y).$$

Variants of strict or uniform displacement convexity can then be studied in the same way as before, taking into account the important identity (5.42), which implies that $\int |v_t|^2\, d\rho_t$ is always equal to $\mathcal{T}_2(\rho_0, \rho_1)$.

5.4.6. Hopf-Lax formula. Generally speaking, (time-dependent) Hamilton-Jacobi equations are equations of the form

$$\frac{\partial u}{\partial t} + H(t, x, u, \nabla u) = 0, \tag{5.53}$$

where H is the so-called **Hamiltonian**. These equations have been intensively studied; an introduction to their mathematical theory can be found in [**133, 193, 29**].

As a general rule, solutions of Hamilton-Jacobi equations are *not necessarily smooth* and *not unique*. The lack of smoothness means that one has to develop a theory for weak solutions, while the lack of uniqueness means that one should single out the relevant solutions among all possible ones.

The classical theory of Hamilton-Jacobi equations treats the case when $H = H(t, x, w, p)$ does not depend on w, and $H(t, x, p)$ is *convex* with respect to the variable p. Then one can introduce the *Lagrangian*, which is the Legendre transform of H with respect to the variable p:

$$L(t, x, v) = H^*(t, x, v) = \sup_{p\in\mathbb{R}^n}\left[p\cdot v - H(t, x, p)\right]. \tag{5.54}$$

The **Bellman principle** yields a particular solution of equation (5.53):

$$u(t, x) = \inf\left\{u_0(y) + \int_0^t L\big(\tau, \xi(\tau), \dot\xi(\tau)\big)\, d\tau; \qquad \xi(t) = x, \quad \xi(0) = y\right\}. \tag{5.55}$$

When $H(t, x, p)$ depends only on p, (5.55) reduces to the **Hopf-Lax formula**:

$$\begin{aligned} u(t, x) &= \inf_{y, z(\tau)}\left[u_0(y) + \int_0^t L(\dot z(\tau))\, d\tau; \quad z(0) = y,\ z(t) = x\right] \\ &= \inf_{y\in\mathbb{R}^n}\left[u_0(y) + tL\left(\frac{x-y}{t}\right)\right]. \end{aligned} \tag{5.56}$$

The most fundamental case is given by the quadratic Hamiltonian,

$$H(p) = \frac{|p|^2}{2}, \qquad L(v) = \frac{|v|^2}{2}.$$

In this case, a solution of the Hamilton-Jacobi equation

$$\frac{\partial u}{\partial t} + \frac{|\nabla u|^2}{2} = 0 \tag{5.57}$$

is given by

$$\begin{aligned} u(t,x) &= \inf_y \left[u_0(y) + \frac{|x-y|^2}{2t} \right] \\ &= \inf_y \left[u_0(y) + \frac{|y|^2}{2t} - \frac{x \cdot y}{t} \right] + \frac{|x|^2}{2t} \\ &= -\frac{1}{t} \sup_y \left[x \cdot y - \left(t u_0(y) + \frac{|y|^2}{2} \right) \right] + \frac{|x|^2}{2t} \\ &= -\frac{1}{t} \left[t u_0 + \frac{|\cdot|^2}{2} \right]^* (x) + \frac{|x|^2}{2t}, \end{aligned}$$

with the superscript $*$ denoting the Legendre transform.

This very simple particular case will exemplify some of the basic considerations. First, recall that the Legendre transform of a smooth function is in general not smooth; hence the solutions of (5.57) cannot be smooth in general.

Next, recall that the Legendre transform is not injective when applied to nonconvex functions. So, let φ and $\widetilde{\varphi}$ be two distinct functions such that $\varphi^* = (\widetilde{\varphi})^*$, let $t_0 > 0$, and set

$$u_0(y) = \frac{1}{t_0} \left(\varphi(y) - \frac{|y|^2}{2} \right), \qquad \widetilde{u_0}(y) = \frac{1}{t_0} \left(\widetilde{\varphi}(y) - \frac{|y|^2}{2} \right).$$

Then the Hopf-Lax formula allows us to construct solutions $u(t,x)$ and $\widetilde{u}(t,x)$ of (5.57) such that $u(t_0,x) = \widetilde{u}(t_0,x)$. Now, note that equation (5.53) is *reversible*: if $u(t,x)$ is a solution of (5.53), then $-u(-t,x)$ also is. Thus, reversing time, we may construct an infinite number of solutions starting from the same initial datum! This shows that one cannot hope for uniqueness.

To remedy these problems, Crandall, Evans and Lions introduced the concept of **viscosity solutions**, which we already encountered in Chapter 4. This notion makes it possible to deal with weak solutions (in general, not better than locally Lipschitz) and at the same time to ensure uniqueness. As a necessary price, reversibility is lost. On the other hand, there is a weak replacement for regularization: when the Hamiltonian H is uniformly convex, then solutions automatically become semiconcave for $t > 0$; this effect is discussed at a very simple level in Evans [**133**, chapter 3.3].

We do not wish to develop more about the properties of viscosity solutions here, and we shall only mention that the Hopf-Lax formula, when it applies, indeed yields the unique viscosity solution.

As a by-product, we shall be able to characterize c-concave functions in terms of Hamilton-Jacobi equations. Recall from Section 2.5 that c-concave functions ψ are defined as

$$\psi(x) = \inf_y \big[c(x-y) + \zeta(y)\big]$$

for some function ζ, which without loss of generality can be assumed to be continuous, and even locally Lipschitz (by the usual "double-c-concavification trick"). Thus, we are led to

Proposition 5.45 (PDE characterization of c-concave functions). *Let c be a convex cost function on $\mathbb{R}^n$. Then, c-concave functions are exactly all viscosity solutions of the Hamilton-Jacobi equation*

$$\frac{\partial u}{\partial t} + c^*(\nabla u) = 0 \tag{5.58}$$

at time $t = 1$, or equivalently at a time $t \geq 1$.

Proof. At this point, the only thing to check is the equivalence between "solving a Hamilton-Jacobi equation at time $t = 1$" and "solving a Hamilton-Jacobi equation at time $t \geq 1$". This is an immediate consequence of the semigroup property enjoyed by viscosity solutions of the Hamilton-Jacobi equation, itself a consequence of the uniqueness property. □

Remark 5.46. The initial datum does not play any role; it can be any locally Lipschitz function on $\mathbb{R}^n$.

Note that the loss of reversibility is essential in (5.45)! A solution of the Hamilton-Jacobi equation at time $t = 1$ cannot be *any* function. By combining Proposition 5.45 with the theory of Hamilton-Jacobi equations, one can sometimes find regularity statements for c-concave functions. One can also recover the following consequence.

Proposition 5.47 (c-concave functions can be rescaled for homogeneous costs). *Assume that c is homogeneous, say $c(z) = |z|^p$, and let ψ be a c-concave function. Then, for all $\lambda \in [0, 1]$, $\lambda\psi$ is also c-concave.*

Proof. Let $u = u(t,x)$ be such that $\partial_t u + c^*(\nabla u) = 0$, $u(1, \cdot) = \psi$. Then consider

$$\widetilde{u}(t,x) = \lambda u(\lambda^{p'-1}t, x).$$

It is immediate to check that $\widetilde{u}$ also satisfies the Hamilton-Jacobi equation with Hamiltonian c^* (here the positivity of λ is important for $\widetilde{u}$ to be a viscosity solution). Choosing $t = 1/(\lambda^{p'-1}) \geq 1$, we see that $\widetilde{u}(t, \cdot)$ is c-concave; but so is $\lambda u(1, \cdot) = \lambda\psi$. □

To conclude this subsection, we rewrite the Kantorovich duality in terms of Hamilton-Jacobi equations. After the above remarks, the following proposition is almost obvious.

Proposition 5.48 (Hamilton-Jacobi formulation of the Kantorovich duality). *Let $c : \mathbb{R}^n \to \mathbb{R}_+$ be a strictly convex, superlinear cost function, and let μ, ν be two probability measures on $\mathbb{R}^n$. Then*

$$\mathcal{T}_c(\mu,\nu) = \sup \left\{ \int_{\mathbb{R}^n} \varphi(1,\cdot)\, d\nu - \int_{\mathbb{R}^n} \varphi(0,\cdot)\, d\mu \right\}, \tag{5.59}$$

where the supremum is taken over all viscosity solutions $\varphi : [0,1] \times \mathbb{R}^n \to \mathbb{R}$ of

$$\begin{cases} \dfrac{\partial \varphi}{\partial t} + c^*(\nabla_x \varphi) = 0, \\ \\ \varphi(0,\cdot) \equiv \varphi_0 \in C_b(\mathbb{R}^n). \end{cases} \tag{5.60}$$

This proposition is an immediate consequence of Theorem 1.3 and the Hopf-Lax formula (5.56) with $t = 1$, $L = c$. We note that, by Theorem 2.44, the supremum in (5.59) is attained as soon as $\mathcal{T}_c(\mu,\nu) < +\infty$, provided that we allow more general initial data in (5.60).

5.4.7. Riemannian manifolds. Again, most of the previous discussion can be transported to the case of a more general Riemannian manifold. Equations (5.46) are certainly the most intrinsic among those that we have seen, and in particular can be generalized to a Riemannian manifold. For instance, in the case of transportation with a quadratic cost, the optimality equations read

$$\begin{cases} \dfrac{\partial \rho}{\partial t} + \nabla \cdot (\rho \nabla u) = 0, \qquad \rho(t=0,\cdot) = \mu, \\ \\ \dfrac{\partial u}{\partial t} + \dfrac{|\nabla u|^2}{2} = 0, \qquad -u(t=0,\cdot)\ \dfrac{d^2}{2}\text{-concave}, \end{cases} \tag{5.61}$$

and the equation on u can be solved by the Hopf-Lax formula, mutatis mutandis.

As for properties of displacement convexity, they can often be studied with the help of the famous **Bochner formula** [**147**, Section 4.15]: whenever u is a smooth function on a smooth manifold M, then

$$-\nabla u \cdot \nabla \Delta u + \Delta \frac{1}{2} |\nabla u|^2 \;=\; \mathrm{tr}\left((D^2 u)^T D^2 u\right) + \langle \mathrm{Ric} \cdot \nabla u, \nabla u \rangle, \tag{5.62}$$

where ∇ stands for the gradient operator on M, $\nabla\cdot$ for the divergence operator, Ric for the Ricci tensor, D^2 for the Hessian operator and Δ for the Laplace-Beltrami operator on M.

In [**219**], the Bochner formula was applied to give a formal proof that the functional $\int \rho \log \rho$ is displacement convex on a manifold with nonnegative Ricci curvature; this was proven rigorously later in [**100**].

The difficulties which we already encountered, linked to boundary conditions and positivity of the interpolant, appear also in this case. In fact, even if one works on a manifold without boundary, and with probability measures μ, ν having positive densities, it is not known whether the time-interpolants $[\mu, \nu]_t$ will also be positive everywhere, i.e. whether the transportation maps will be surjective.

5.4.8. A remark on injectivity. Usually, the study of Hamilton-Jacobi equations requires a lot of care, due to shocks, which are encounters between trajectories. Indeed, except for very peculiar profiles of initial velocities v_0, the characteristic lines $x - tv_0(x)$ will always join at some point and some time $t > 0$, and at that moment the Lagrangian interpretation of (5.39) is lost. This is why the Lagrangian description usually breaks down in finite time. But here, we are considering Hamilton-Jacobi equations in a very specific context! Due to the particular structure of the initial datum, for a quadratic cost function *there will be no shocks* on the time interval which is of interest for us, i.e. $(0, 1)$. This can be seen at the level of the explicit Lagrangian solution: assume that there is a shock,

$$(1-t)x + t\nabla\varphi(x) = (1-t)y + t\nabla\varphi(y), \qquad x \neq y,$$

where φ is the convex Kantorovich potential. Then, from the equation and the convexity of φ

$$0 \leq \big\langle \nabla\varphi(x) - \nabla\varphi(y), x - y \big\rangle = -\frac{(1-t)}{t}|x-y|^2 \leq 0,$$

which is possible only if $t = 1$. Thus, there are no problems of continuation of characteristics in $(0, 1)$.

Things are more tricky for nonquadratic cost functions, but it remains true that there are no shocks for strictly convex cost functions: see [**98**, Proposition 5.1]. In the Riemannian case, similar results is known at least for the quadratic cost: in this situation also, there is no shock, and the optimal transportation maps $(T_t)_{0\leq t\leq 1}$ are injective.

Theorem 5.49 (Injectivity theorem on a Riemannian manifold). *Let μ, ν be two absolutely continuous, compactly supported probability measures on a smooth complete Riemannian manifold, and let $\nabla\psi$ be the unique gradient of a $d^2/2$-concave function such that $\exp_x(-\nabla\psi)\#\mu = \nu$. Then, for all $t \in (0, 1)$, $x \mapsto \exp_x(-t\nabla\psi(x))$ is injective.*

The following proof, suggested by McCann, is amazingly simple and robust:

Proof. Let x, x', y, y' be such that the optimal transportation maps x onto x', y onto y', and the geodesics going from x to x', and from y to y' respectively, do cross at some time $t \in (0,1)$. In other words, there exists $m \in M$ such that

$$(5.63) \qquad \begin{cases} d(x,m) = t\, d(x,x'), & d(m,x') = (1-t)\, d(x,x'), \\ d(y,m) = t\, d(y,y'), & d(m,y') = (1-t)\, d(y,y'). \end{cases}$$

Write down the two triangle inequalities,

$$d(x,y') \le d(x,m) + d(m,y'), \qquad d(x',y) \le d(x',m) + d(m,y),$$

and square them up to get

$$\begin{aligned} d(x,y')^2 + d(x',y)^2 \le d(x,m)^2 + d(x',m)^2 + d(x,y')^2 + d(m,y)^2 \\ + 2\, d(x,m)\, d(y',m) + 2\, d(y,m)\, d(x',m). \end{aligned}$$

Taking into account (5.63), this last inequality can be rewritten as

$$d(x,y')^2 + d(x',y)^2 \le d(x,x')^2 + d(y,y')^2 - 2t(1-t)[d(x,x') - d(y,y')]^2$$

(check!). Hence,

$$d(x,y')^2 + d(x',y)^2 \le d(x,x')^2 + d(y,y')^2,$$

and one easily sees that equality cannot occur if the geodesic lines $[x,x']$ and $[y,y']$ are distinct. In other words, a shock would imply

$$d(x,y')^2 + d(x',y)^2 < d(x,x')^2 + d(y,y')^2.$$

On the other hand, recall from Chapter 2 (see Remark 2.26 (i) or Exercise 2.38) that the support of the optimal transference plan lies in a $d^2/2$-cyclically monotone set, so for $d\mu$-almost all x, x' and $d\nu$-almost all y, y',

$$d(x,y')^2 + d(x',y)^2 \ge d(x,x')^2 + d(y,y')^2.$$

We conclude that the crossing of geodesics $[x,x']$ and $[y,y']$ is an event of zero probability. □

Remark 5.50. The injectivity condition means: $\forall t \in (0,1)$, $\exp_x(-t\nabla\psi(x))$ is an injective map. It *does not* mean that trajectories, in the geometric sense (considered as subsets of M), cannot cross — as demonstrated by Problem 1 in Chapter 10, they can!

5.4.9. Rigorous justification. Let us present a rigorous way to implement the above formal considerations. Our discussion here will be restricted to the case of the quadratic cost in $\mathbb{R}^n$. One advantage of this case is that it can be treated without any knowledge of the theory of Hamilton-Jacobi equations. For a more general homogeneous cost function, the behavior of the cost near 0 should be taken into account carefully. On a manifold the corresponding study has not yet been performed, although this would surely be of interest.

Theorem 5.51 (Displacement interpolation in Eulerian formulation). *Let ρ_0 and ρ_1 be two probability measures in $P_{\mathrm{ac},2}(\mathbb{R}^n)$, and let $\nabla\varphi$ be a gradient of a convex function such that $\nabla\varphi\#\rho_0 = \rho_1$. Let $u_0(x) = \varphi(x) - |x|^2/2$, and for $0 < t \le 1$ let u_t be defined by the Hopf-Lax formula*

$$u_t(x) = \inf_{y\in\mathbb{R}^n}\left[u_0(y) + \frac{|x-y|^2}{2t}\right].$$

Moreover, let $\rho_t = T_t\#\rho_0$ be McCann's interpolant between ρ_0 and ρ_1, where $T_t = t\nabla\varphi + (1-t)\mathrm{Id}$. Then,

(i) For all $t \in (0,1)$, u_t is locally Lipschitz, and its gradient $v_t = \nabla u_t$, defined almost everywhere, coincides with $\nabla u_0 \circ T_t^{-1}$ on $T_t(\mathbb{R}^n)$. Moreover, v itself is locally Lipschitz in t and x on $T_t(\mathbb{R}^n)$. More precisely, for all $\varepsilon > 0$, on the image of $[\varepsilon, 1-\varepsilon] \times \mathbb{R}^n$ under the mapping $(t,x) \longmapsto (t, T_t(x))$, v is Lipschitz in x (uniformly in t), and Lipschitz in t (locally in x).

(ii) $(\rho_t)_{0<t<1}$ satisfies the linear transport equation

$$\frac{\partial\rho_t}{\partial t} + \nabla\cdot(\rho_t v_t) = 0$$

in weak form on $(0,1)\times\mathbb{R}^n$.

Before giving the proof of the theorem, we explain why v_t should be Lipschitz away from $t = 0$ and $t = 1$; this is an argument which we learnt from Otto. It is a general feature of Hamilton-Jacobi equations with a strictly convex Hamiltonian, that their solutions become semi-concave for positive times, so D^2u_t should be bounded from above for $t > 0$. But if we reverse time and consider the optimal time-dependent transportation problem from ρ_1 to ρ_0, then the (unique) solution is given by reversing all velocities, and therefore changing u_t into $-u_t$. So $-D^2u_t$ should also be bounded from above for $t < 1$. We conclude that D^2u_t should be bounded for $0 < t < 1$, which means that ∇u_t should be Lipschitz.

Proof of Theorem 5.51. As we said before, the solution of the Eulerian problem will be constructed from the solution of the Lagrangian one.

1. First of all, the Hopf-Lax equation can be rewritten as

$$\frac{|x|^2}{2} - tu_t(x) = \sup_{y\in\mathbb{R}^n} \left\{ x \cdot y - \left[\frac{|y|^2}{2} + tu_0(y) \right] \right\}. \tag{5.64}$$

Of course the right-hand side is the Legendre transform of the function $\psi_t : y \longmapsto |y|^2/2 + tu_0(y) = (1-t)|y|^2/2 + t\varphi(y)$. Since ψ_t is uniformly convex, its Legendre transform is differentiable everywhere, and we can write $\nabla\psi_t^* \circ \nabla\psi_t = \mathrm{Id}$. From (5.64), $\nabla\psi_t^* = \mathrm{Id} - tv_t$; so

$$(\mathrm{Id} - tv_t) \circ (\mathrm{Id} + tv_0) = \mathrm{Id},$$

which simplifies into

$$v_t \circ (\mathrm{Id} + tv_0) = v_0.$$

But $\mathrm{Id} + tv_0$ is nothing but T_t. So we have $v_t = v_0 \circ T_t^{-1}$ on $T_t(\mathbb{R}^n)$.

2. Let $x' = T_t(x)$, $y' = T_t(y)$. By direct calculation,

$$\begin{aligned}\big\langle v_t(x') - v_t(y'), x' - y' \big\rangle = -(1-t)|x-y|^2 + t|\nabla\varphi(x) - \nabla\varphi(y)|^2 \\ + (1-2t)\big\langle \nabla\varphi(x) - \nabla\varphi(y), x - y \big\rangle\end{aligned} \tag{5.65}$$

and

$$\begin{aligned}|x' - y'|^2 = (1-t)|x-y|^2 + t^2|\nabla\varphi(x) - \nabla\varphi(y)|^2 \\ + 2t(1-t)\big\langle \nabla\varphi(x) - \nabla\varphi(y), x - y \big\rangle.\end{aligned} \tag{5.66}$$

Combining (5.65) and (5.66), one easily finds that

$$-\frac{|x'-y'|^2}{1-t} \le \big\langle v_t(x') - v_t(y'), x' - y' \big\rangle \le \frac{|x'-y'|^2}{t}.$$

Since v_t is of the form ∇u_t, this implies that the function u_t is both semi-convex and semi-concave:

$$-\frac{I_n}{1-t} \le D^2 u_t \le \frac{I_n}{t}.$$

3. From the equation satisfied by v_t we deduce that it is Lipschitz with respect to $t \in (\varepsilon, 1-\varepsilon)$. Indeed, let t and s be in $(0,1)$, and let $x' = (1-t)x + t\nabla\varphi(x) = x + tv_0(x)$ be in $T_t(\mathbb{R}^n)$; then

$$|v_t(x') - v_s(x')| = |v_s(x + sv_0(x)) - v_s(x + tv_0(x))| \le \|v_s\|_{\mathrm{Lip}} |s-t| |v_0(x)|.$$

If we restrict to $\varepsilon < s, t < 1-\varepsilon$, then $\|v_s\|_{\mathrm{Lip}} \le C_\varepsilon = \max(\varepsilon^{-1}, (1-\varepsilon)^{-1})$. On the other hand, we can write $v_0 = v_t(x')$, so

$$|v_t(x') - v_s(x')| \le C_\varepsilon |s-t| |v_t(x')|. \tag{5.67}$$

If x_0 is an arbitrary point in the interior of the domain of $\nabla\varphi$, then $x_0' = (1-t)x_0 + t\nabla\varphi(x_0)$ stays in a bounded domain of $\mathbb{R}^n$ as t varies in $(0,1)$, and $v_t(x_0') = \nabla\varphi(x_0) - x_0$ is constant; since v_t has Lipschitz norm bounded by C_ε, we deduce that v_t is locally bounded on the image of $[\varepsilon, 1-\varepsilon] \times \mathbb{R}^n$

by $(t, x) \longmapsto (t, T_t(x)) = (t, x')$. Then, from (5.67) we conclude that v_t is a Lipschitz function of t, locally in x.

4. Since $v_t \circ T_t = v_0$, we have

$$\int |v_t|^2 \, d\rho_t = \int |v_0|^2 \, d\rho_0 < +\infty.$$

In particular, $v_t \in L^\infty((0,1); L^2(d\rho_t))$, and the linear transport equation of (ii) makes sense, in the form

$$\frac{d}{dt} \int \zeta \, d\rho_t = - \int v_t \cdot \nabla \zeta \, d\rho_t,$$

which should be satisfied for any smooth test function $\zeta(x)$ with compact support, and almost all t. To prove that this equation holds true, it suffices to copy the proof of Theorem 5.34, taking into account the identities $v_t \circ T_t = v_0 = (d/dt)T_t$ and $T_t \# \rho_0 = \rho_t$. □

Exercise 5.52 (Momentum is transported along optimal paths). Show that $v_t \rho_t = T_t \# (v_0 \rho_0)$ in the sense of vector-valued measures.

Chapter 6

Geometric and Gaussian Inequalities

In this chapter, we present some applications of optimal transportation to the field of functional inequalities with geometrical content. As was implicit in the last chapter for instance, the solution of the optimal transportation problem carries all the information on the geometry of the ambient space, and it is no wonder that it can be applied to problems which involve both functional analysis and geometry. We shall illustrate this on three problems. The first one is a simple proof of the Brunn-Minkowski inequality and its functional counterpart, the Prékopa-Leindler inequality. The second one is a far-reaching generalization, due to Barthe, which implies as particular cases the identification of the optimal constants in the Young inequality for convolution. The third one is a treatment of sharp Sobolev inequalities, taken from work by Cordero-Erausquin, Nazaret and the author.

In all this chapter, the only transportation cost which will be considered is the *quadratic* transportation cost. In fact, in many cases, which transportation cost is used has no influence on the methods of proof, so it is better to stick to the quadratic case, which is the simplest from the technical point of view. Also, many of the arguments only use the existence of a "well-behaved", in some sense, transportation mapping, independently of any optimality property.

The use of mass transportation for such inequalities is not new, since mass transportation appears in a more or less explicit way behind most known proofs of the Brunn-Minkowski inequality. In this context, mass transportation methods are often called **reparametrization techniques**.

However, in some situations optimal transportation does have some advantages; this will be especially true in sections 6.2 and 6.3.

We shall first briefly recall some basic facts about isoperimetry, and about Brunn-Minkowski and Prékopa-Leindler inequalities. The reader who wants to know more may consult Burago and Zalgaller [**69**] and Osserman [**213**] for a detailed account on isoperimetric problems, and Gardner [**152**] and Schneider [**234**] for the Brunn-Minkowski theory. On the whole, these sources contain about two thousand references! Particularly recommended is Gardner's survey paper [**152**], which provides an up-to-date and enthusiastic account of the field and some of its connections to such diverse areas as algebraic geometry, combinatorics, etc. A concise and synthetic account on the particular topics that we shall develop can also be found in the PhD thesis of Barthe [**30**].

6.1. Brunn-Minkowski and Prékopa-Leindler inequalities

6.1.1. Isoperimetry. Everybody knows the solution of the Euclidean isoperimetric problem: among all compact sets in $\mathbb{R}^n$ with given volume, the sphere has minimal surface. There are many proofs of this theorem (based on symmetrization, for instance). A short way to get it, very analytic in spirit, is via the fundamental **Brunn-Minkowski inequality** (first stated by Brunn in 1887 for $n = 3$): given any two compact sets X and Y in $\mathbb{R}^n$,

$$|X+Y|^{\frac{1}{n}} \geq |X|^{\frac{1}{n}} + |Y|^{\frac{1}{n}}. \tag{6.1}$$

Here $|X|$ stands for the Lebesgue measure of X, so $|X|^{1/n}$ is a kind of typical length for X. Moreover, $X + Y$ is the **Minkowski sum** of X and Y:

$$X + Y = \{x + y; \quad x \in X,\, y \in Y\}.$$

Since $|\lambda X|^{1/n} = |\lambda||X|^{1/n}$ whenever $\lambda \in \mathbb{R}$, equation (6.1) can also be rewritten as

$$\left|\frac{X+Y}{2}\right|^{\frac{1}{n}} \geq \frac{1}{2}\left(|X|^{\frac{1}{n}} + |Y|^{\frac{1}{n}}\right).$$

Thus, it expresses the concavity of $|\cdot|^{1/n}$ with respect to the operation of Minkowski sum.

Remark 6.1. The assumption of compactness of X and Y ensures the compactness of $X + Y$ and avoids the delicate problems linked to the measurability of the Minkowski sum.

To see how (6.1) implies the isoperimetric statement, let us introduce the unit ball, $B = B(0, 1)$, centered at 0 and with unit radius, and let us

denote by $B_r = rB$ the ball of radius $r > 0$. We consider a small positive number ε, and choose $Y = B_\varepsilon$ in (6.1), which can be rewritten as

$$\frac{|X + B_\varepsilon|^{\frac{1}{n}} - |X|^{\frac{1}{n}}}{\varepsilon} \geq |B|^{\frac{1}{n}}. \tag{6.2}$$

By definition, the surface of X is

$$S(X) = \liminf_{\varepsilon \downarrow 0} \frac{|X + B_\varepsilon| - |X|}{\varepsilon}. \tag{6.3}$$

Thus from (6.2) we deduce, by passing to the lim inf,

$$\frac{1}{n}|X|^{\frac{1}{n}-1} S(X) \geq |B|^{\frac{1}{n}}.$$

If $|X| = |B_r| = r^n|B|$, then this last inequality can be rewritten as

$$S(X) \geq n|B|r^{n-1} = S(B_r)$$

(recall that $S(B) = n|B|$); or

$$\left[\frac{S(X)}{S(B)}\right]^{\frac{1}{n-1}} \geq \left(\frac{|X|}{|B|}\right)^{\frac{1}{n}}. \tag{6.4}$$

This is the isoperimetric statement.

Refined versions of the Brunn-Minkowski inequality are used in the study of many isoperimetric-type inequalities, in relation with the **Wulff shape** for instance (see [**144**]).

Remark 6.2. Definition 6.3 coincides with other possible definitions of surface (like $\|\nabla(1_X)\|_{TV}$) when X is regular enough. If this is not the case, a regularization argument can still be applied to recover the isoperimetric inequality for any reasonable definition of surface. Also, it is possible to state the Euclidean isoperimetry inequality without using the concept of surface, as follows: let $A \subset \mathbb{R}^n$ be a measurable set with finite Lebesgue measure (or volume) $|A|$, let B be the Euclidean ball with the same volume as A, and let A^t denote the set of all points in $\mathbb{R}^n$ whose distance to A is at most t. Then, we have the inequality

$$|A^t| \geq |B^t|. \tag{6.5}$$

By the way, there is a version of this inequality on the sphere S^{n-1} (where balls are replaced by spherical caps), which has been used in a rather surprising way by Dudley [**125**] to show that the Kantorovich duality does not extend to a nonmeasurable setting. See [**125**, p. 184] for background and bibliography about this inequality and its relation to the isoperimetric inequality on the sphere.

6.1.2. Proof of Brunn-Minkowski by mass transportation. As was understood by McCann, the Brunn-Minkowski inequality can be seen as a consequence of the displacement convexity of the functional

$$\mathcal{U}(\rho) = -\int_{\mathbb{R}^n} \rho(x)^{1-\frac{1}{n}}\, dx.$$

This choice of $\mathcal{U}$ is easy to understand: if μ is the uniform probability measure carried by X,

$$d\mu(x) = \frac{1_X\, dx}{|X|},$$

then $\mathcal{U}(\mu) = -|X|^{1/n}$. The link with the Minkowski sum appears by the following easy lemma.

Lemma 6.3 (The interpolant is supported in the Minkowski sum). *Let $\mu = \rho_0$, $\nu = \rho_1$ be the uniform probability measures carried by the compact sets X and Y. Then, for all $t \in [0,1]$, the interpolant $\rho_t = [\mu,\nu]_t$ has its support included in the Minkowski combination $(1-t)X + tY$.*

The proof of this lemma is left as an exercise. We note that the inclusion is in general strict.

Proof of (6.1). Let $t \in (0,1)$, and let S_t denote the support of ρ_t. Of course $1_{S_t} dx/|S_t|$ is a probability measure; therefore, by Jensen's inequality,

$$\mathcal{U}(\rho_t) = \int_{S_t} U\left(\frac{d\rho_t}{dx}\right) dx \geq |S_t| U\left(\frac{1}{|S_t|}\int d\rho_t\right) = |S_t| U\left(\frac{1}{|S_t|}\right) = -|S_t|^{\frac{1}{n}}.$$

But by Lemma 6.3,

$$-|S_t|^{\frac{1}{n}} \geq -|(1-t)X + tY|^{\frac{1}{n}}.$$

Therefore the displacement convexity of $\mathcal{U}$ (Theorem 5.15), namely

$$\mathcal{U}(\rho_t) \leq (1-t)\mathcal{U}(\rho_0) + t\mathcal{U}(\rho_1),$$

implies

$$\big|(1-t)X + tY\big|^{\frac{1}{n}} \geq (1-t)|X|^{\frac{1}{n}} + t|Y|^{\frac{1}{n}},$$

which is equivalent to (6.1). □

6.1.3. The Prékopa-Leindler inequality. There is a *functional version* of the Brunn-Minkowski inequality traditionally called the Prékopa-Leindler inequality [**224, 185**], and actually proven many times in various forms, by various authors, back to the fifties. It can be expressed as follows.

Theorem 6.4 (Prékopa-Leindler inequality). *Let f, g, h be three non-negative integrable functions on $\mathbb{R}^n$, and $\lambda \in [0,1]$. Assume that for* all *$x, y \in \mathbb{R}^n$,*

$$h\big((1-\lambda)x + \lambda y\big) \geq f(x)^{1-\lambda} g(y)^{\lambda}.$$

Then,

$$\int_{\mathbb{R}^n} h \geq \left(\int_{\mathbb{R}^n} f\right)^{1-\lambda} \left(\int_{\mathbb{R}^n} g\right)^{\lambda}. \tag{6.6}$$

Remarks 6.5. (i) Of course, one may choose

$$h(z) = \sup_{z=(1-\lambda)x+\lambda y} f(x)^{1-\lambda} g(y)^{\lambda}.$$

(ii) Contrary to what was written in the first edition of this book, one may do it with the usual supremum replaced by an essential supremum; see Brascamp and Lieb [**57**]. The simple proof there is based on the Brunn-Minkowski inequality [**152**].

Theorem 6.4 is actually equivalent to the Brunn-Minkowski inequality, in the sense that one can go very easily from one statement to the other. For instance, to recover (6.1) from (6.6), it is sufficient to choose f and g respectively equal to the characteristic functions of $X/(1-\lambda)$ and Y/λ, thus obtaining

$$|X+Y| \geq \left|\frac{X}{\lambda}\right|^{\lambda} \left|\frac{Y}{1-\lambda}\right|^{1-\lambda};$$

maximizing the right-hand side with respect to λ amounts to choosing $\lambda = |Y|^{1/n}/(|x|^{1/n} + |Y|^{1/n})$, and this results in

$$|X+Y| \geq (|X|^{1/n} + |Y|^{1/n})^n.$$

On the other hand, the functional character of (6.4) gives it a lot of flexibility. This theorem is commonly used in many geometrical applications.

Since (6.1) and (6.6) are equivalent, we already have a proof of the Prékopa-Leindler inequality via optimal transportation. However, it is interesting to also have a direct proof of Theorem 6.4. We shall give two proofs, which are slightly different, but both rely on optimal transportation. The first argument is due to McCann [**201**], the second one to Barthe [**30**]. We also mention that an old and classical proof of (6.1) by Knothe (see [**234**, p. 312]) is actually a proof by transportation — but not optimal transportation.

6.1.4. First proof of the Prékopa-Leindler inequality. This proof is exctracted from the PhD thesis of McCann [**201**]. Without loss of generality, we may assume that $\int f = \int g = 1$ in (6.6). So we just have to prove that $\int h \geq 1$. Identify the functions f and g with the corresponding probability measures, and consider $(\rho_t)_{0\leq t\leq 1}$, the associated displacement interpolant between f and g. Of course $\int \rho_t = 1$. So (6.6) will follow once we know that

$$0 \leq \lambda \leq 1 \implies \rho_\lambda \leq h \quad \text{almost everywhere.} \tag{6.7}$$

To prove (6.7), we use the Monge-Ampère equation. Let φ be a convex function such that $\nabla\varphi\#f = g$. Recall from the definition of displacement interpolation, and from Theorem 4.8, that the Monge-Ampère equation

$$f(x) = \rho_\lambda\big((1-\lambda)x + \lambda\nabla\varphi(x)\big)\,\det\big((1-\lambda)I_n + \lambda D_A^2\varphi(x)\big) \tag{6.8}$$

is satisfied almost everywhere, with D_A^2 standing for the Aleksandrov second derivative. In particular,

$$f(x) = g(\nabla\varphi(x))\,\det(D_A^2\varphi(x)).$$

It follows that $g(\nabla\varphi(x)) \neq 0$ for $f(x)\,dx$-almost all x, so $\det D_A^2\varphi(x) = f(x)/g(\nabla\varphi(x))$, for $f(x)\,dx$-almost all x. The idea is now to express the Jacobian determinants in terms of f and g. By Lemma 5.23,

$$\begin{aligned}\det\Big((1-\lambda)I_n + \lambda D_A^2\varphi(x)\Big)^{\frac{1}{n}} &\geq (1-\lambda)(\det I_n)^{\frac{1}{n}} + \lambda(\det D_A^2\varphi)^{\frac{1}{n}}\\ &= (1-\lambda) + \lambda\left(\frac{f(x)}{g(\nabla\varphi(x))}\right)^{\frac{1}{n}}.\end{aligned}$$

Inserting this in (6.8), one finds that

$$\begin{aligned}\rho_\lambda\Big((1-\lambda)x + \lambda\nabla\varphi(x)\Big) &\leq \frac{f(x)}{\left((1-\lambda) + \lambda\left(\frac{f(x)}{g(\nabla\varphi(x))}\right)^{\frac{1}{n}}\right)^n}\\ &= \Big[(1-\lambda)f(x)^{-\frac{1}{n}} + \lambda g(\nabla\varphi(x))^{-\frac{1}{n}}\Big]^{-n}\\ &\leq f(x)^{1-\lambda}g(\nabla\varphi(x))^\lambda,\end{aligned}$$

where the last inequality is again the arithmetic-geometric inequality from Lemma 5.23. This inequality holds true for $f(x)\,dx$-almost all x, but both sides are 0 when $f(x) = 0$, so the inequality really holds almost everywhere.

As a conclusion, for almost all $x \in \mathbb{R}^n$,

$$\rho_\lambda((1-\lambda)x + \lambda\nabla\varphi(x)) \leq f(x)^{1-\lambda}g(\nabla\varphi(x))^\lambda \leq h\big((1-\lambda)x + \lambda\nabla\varphi(x)\big).$$

Since $[(1-\lambda)\mathrm{Id} + \lambda\nabla\varphi](\mathbb{R}^n)$ is of full measure for ρ_λ, this implies (6.7). □

Remarks 6.6. (i) The Monge-Ampère equation is satisfied in the classical sense if f and g are positive smooth functions on a bounded convex set, and one can always reduce to this case by a regularization procedure. This line of reasoning has however the drawback of relying on the sophisticated Caffarelli regularity theory.

(ii) As a general fact, the displacement interpolant between probability measures supported by X and Y has a support which is strictly smaller than the Minkowski combination. This is usually a drawback, and may be avoided by the strategy of the following subsection. This strategy is also useful for the solution of the problems in Sections 6.2 and 6.3 below.

6.1.5. Second proof of the Prékopa-Leindler inequality. This proof is extracted from Barthe [**30**]. The main idea is the following: instead of interpolating between the two probability densities f and g, we shall introduce another *arbitrary* probability measure, say p, and consider a linear interpolation between the optimal transportation maps from f to p on one hand, from g to p on the other hand.

For instance, we define p to be the Lebesgue density on $[0,1]^n$, and we introduce the optimal transportation maps $\nabla\varphi_1$ transporting p onto f and $\nabla\varphi_2$ transporting p onto g. We can write the Monge-Ampère equations

$$f(\nabla\varphi_1(x))\,\det(D_A^2\varphi_1(x)) = 1, \qquad g(\nabla\varphi_2(x))\,\det(D_A^2\varphi_2(x)) = 1,$$

for almost all $x \in [0,1]^n$. Define $\varphi = (1-\lambda)\varphi_1 + \lambda\varphi_2$. Then

$$\begin{aligned}\int_{\mathbb{R}^n} h &\geq \int_{[0,1]^n} h(\nabla\varphi(x))\det D_A^2\varphi(x)\,dx\\ &\geq \int_{[0,1]^n} h\Big((1-\lambda)\nabla\varphi_1 + \lambda\nabla\varphi_2\Big)\,(\det D_A^2\varphi_1)^{1-\lambda}(\det D_A^2\varphi_2)^{\lambda}\\ &\geq \int_{[0,1]^n} f(\nabla\varphi_1)^{1-\lambda}g(\nabla\varphi_2)^{\lambda}\,(\det D_A^2\varphi_1)^{1-\lambda}(\det D_A^2\varphi_2)^{\lambda}\\ &= \int_{[0,1]^n} 1 = 1,\end{aligned}$$

which proves Theorem 6.4. □

Remark 6.7. In the above computation we used part (iii) of Lemma 5.23. That inequality is sometimes called "**Prékopa-Leindler inequality for Gaussians**", because this is precisely what comes out when one applies the Prékopa-Leindler inequality to Gaussian functions, i.e. exponentials of (negative definite) quadratic forms. Hence, in some sense the general version of the Prékopa-Leindler inequality can be seen as a consequence of the particular Gaussian case. We shall see later that this is no accident.

6.1.6. Generalizations. Several generalizations of the Prékopa-Leindler inequality can be proven in pretty much the same way as above. In particular, the following family of inequalities was obtained for $\alpha > 0$ by Henstock and McBeath, and in the general case by Borell on one hand, Brascamp and Lieb on the other. It involves a generalization of the arithmetic mean: for any two nonnegative numbers a, b, define

$$M_\alpha^\lambda(a,b) = \begin{cases} [\lambda a^\alpha + (1-\lambda)b^\alpha]^{1/\alpha} & \text{if } a,b>0;\\ 0 & \text{else.}\end{cases}$$

Then we have

Theorem 6.8 (Henstock-McBeath inequalities). *Let f, g, h, be three nonnegative integrable functions on $\mathbb{R}^n$, and $\lambda \in [0,1]$, $\alpha \geq -1/n$. Assume that for all $x, y \in \mathbb{R}^n$,*

$$h\big(\lambda x + (1-\lambda)y\big) \geq M_\alpha^\lambda\,[f(x), g(y)].$$

Then,

$$\int_{\mathbb{R}^n} h \geq M^\lambda_{\frac{\alpha}{1+n\alpha}}\left(\int_{\mathbb{R}^n} f, \int_{\mathbb{R}^n} g\right).$$

This theorem can be proven in pretty much the same way as Theorem 6.4 provided that one replaces the arithmetic-geometric inequality by the following consequence of the Hölder inequality: whenever $\alpha + \beta \geq 0$, $1/\alpha + 1/\beta = 1/\gamma$, then

$$M_\alpha{}^\lambda(a,b)\, M_\beta{}^\lambda(c,d) \geq M_\gamma^\lambda(ac, bd).$$

6.1.7. Riemannian manifolds. To conclude this section, we mention the beautiful work accomplished by Cordero-Erausquin, McCann and Schmuckenschläger [**100**] to generalize the proofs above to Riemannian manifolds, and establish some analogs to the Prékopa-Leindler inequalities in this setting. Of course, the geometry of the space plays a crucial role, via Ricci curvature for instance. Many new technical difficulties arise in this study.

6.2. The Alesker-Dar-Milman diffeomorphism

In the preceding section we have seen some links between mass transportation and the Minkowski sum. It is natural to expect that the tools of mass transportation can help describe the Minkowski sum $X + Y$ of two subsets X, Y of $\mathbb{R}^n$; however, in full generality this problem is much too complicated. First of all, the topology of the sets may change: as an exercise, the reader can construct two compact sets X and Y, simply connected, such that $X + Y$ is not simply connected.

To avoid such problems, we may restrict our study to the case when X and Y are convex; but even in this case, there cannot be any general answer. Consider for instance the case when $X = Y = B_1$ (the unit ball); then $X + Y = B_2$, which is clearly diffeomorphic to X. On the contrary, consider, in $\mathbb{R}^2$, the two segments $X = [(0,0),(0,1)]$, $Y = [(0,0),(1,0)]$: then $X + Y = [0,1]^2$, which is extremely different from X and Y. In the first example, a lot of information was lost in the process of Minkowski summation: think that

$$B_2 = \{x + y; \quad x \in B_1,\ y \in B_1\},$$

but also

$$B_2 = \{x + x; \quad x \in B_1\},$$

though one may have a priori expected the latter sum to be much smaller than the former one! In the second example, on the contrary, x and y are uniquely determined by the relation $x + y = z$.

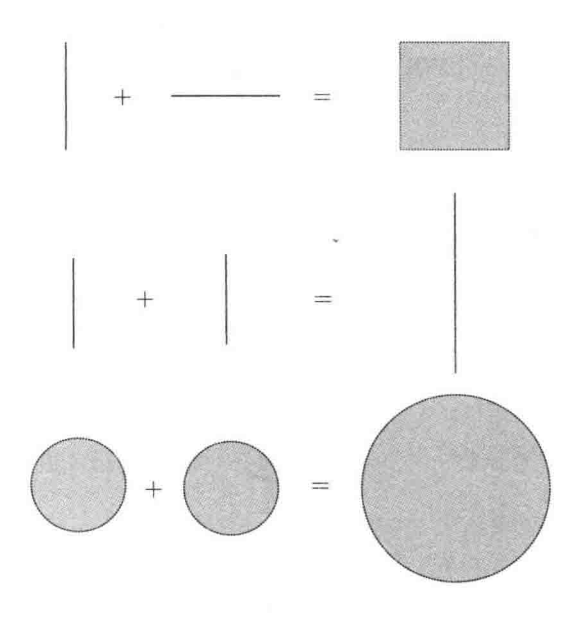


Figure 6.1. The strange rules of Minkowski addition

The following theorem of Alesker, Dar and Milman [**10**] states that, when X and Y are bounded convex open sets, then the situation exemplified above by the case of the ball is the rule: *the Minkowski sum can be parametrized by either of the two bodies.* Let us give a precise statement.

Theorem 6.9 (Alesker-Dar-Milman theorem). *Let X and Y be two open convex bounded subsets of $\mathbb{R}^n$ with volume 1. Then, there exists a C^1-diffeomorphism $\Psi : X \to Y$, preserving the Lebesgue measure, such that for every $\lambda > 0$,*

$$X + \lambda Y = \{x + \lambda\Psi(x), \; x \in X\}.$$

Sketch of proof of Theorem 6.9. The strategy is reminiscent of one of the arguments that we used to prove the Prékopa-Leindler inequality in the previous section. Let μ, ν be the uniform probability measures on X, Y respectively. Consider on $\mathbb{R}^n$ an arbitrary probability measure p with smooth positive density. Let $\nabla\varphi_1$ and $\nabla\varphi_2$ be the optimal transportations from p to μ, ν respectively. Recall that $\nabla\varphi_1$ (resp. $\nabla\varphi_2$) is almost everywhere invertible on X (resp. Y). Moreover, the convex functions φ_1 and φ_2 are of class C^2 by Caffarelli's regularity theory (Theorem 4.14), with strictly

positive Hessian. This is sufficient to imply (see [**159**] for instance) that for any $\lambda > 0$,

$$(\nabla\varphi_1 + \lambda\nabla\varphi_2)(\mathbb{R}^n) = \nabla\varphi_1(\mathbb{R}^n) + \lambda\,\nabla\varphi_2(\mathbb{R}^n) = X + \lambda Y.$$

Therefore it suffices to choose

$$\Psi = \nabla\varphi_2 \circ (\nabla\varphi_1)^{-1}.$$

□

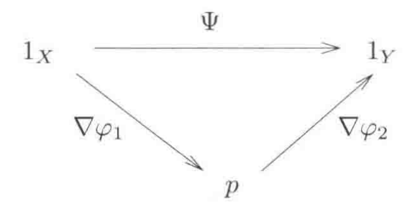


Figure 6.2. Construction of the Alesker-Dar-Milman diffeomorphism

As an application of this result, Alesker, Dar and Milman present a new proof of some of the famous **Aleksandrov-Fenchel inequalities** for mixed volumes: let $K_1, \ldots, K_n$ be n convex subsets of $\mathbb{R}^n$; then

$$V(K_1, \ldots, K_n) \geq \left(\prod_{i=1}^{n} |K_i|\right)^{\frac{1}{n}},$$

where $|K_i|$ stands as usual for the volume of $|K_i|$, and $V(K_1, \ldots, K_n)$ stands for the **mixed volume** of $K_1, \ldots, K_n$. We recall the definition of the mixed volume $V(K_1, \ldots, K_n)$: it is the coefficient of the monomial in $t_1 \ldots t_n$ in the polynomial

$$P(t_1, \ldots, t_n) = |t_1 K_1 + \ldots + t_n K_n|. \tag{6.9}$$

We refer to [**10**] for a proof based on Theorem 6.9, which is simple but sharp enough to yield equality cases.

Exercise 6.10. It is not at all obvious at first sight that the expression in (6.9) is a polynomial! Prove that it is indeed, using optimal transportation, and the following fact: whenever $M_1, \ldots, M_n$ are nonnegative matrices, then $\det(t_1 M_1 + \ldots + t_n M_n)$ is a homogeneous polynomial in $t_1, \ldots, t_n$.

Remark 6.11. This strategy does not so far lead to the whole family of Aleksandrov-Fenchel inequalities (see [**234**]). A possible way to recover the remaining cases would be via the study of Hessian equations, generalizing the Monge-Ampère equations, as studied for instance by Urbas [**259, 260**].

6.3. Gaussian inequalities

In this section we turn to another application, which is also tightly linked to geometry, but whose consequences are on the analytic side. What we call "Gaussian inequalities" are certain families of inequalities in which Gaussian densities play a crucial role (e.g. for cases of equality). Let us recall the definition of a Gaussian.

Definition 6.12 (Gaussian densities). A Gaussian density on $\mathbb{R}^n$ is a function of the form

$$\gamma(x) = \gamma_0 \, \frac{\exp\left(-\frac{1}{2}\langle A^{-1}(x-x_0),\, x-x_0\rangle\right)}{(2\pi)^{n/2}(\det A)^{1/2}},$$

where γ_0 is a nonnegative constant, x_0 is a vector in $\mathbb{R}^n$ and A is a positive definite symmetric matrix. This density is said to be centered if $x_0 = 0$.

Remark 6.13. With the above conventions, $\int \gamma = \gamma_0$. So when $\gamma_0 = 1$, the above density γ is a probability density, with mean x_0 and covariance matrix A. A random variable having a Gaussian distribution is called Gaussian. One can generalize the definition of Gaussian densities to allow singular matrices, in which case γ is concentrated on an affine subspace of $\mathbb{R}^n$ having dimension strictly less than n. With this extended definition, Gaussian random variables possess the remarkable property that any affine image of a Gaussian random variable is still Gaussian. This makes it likely that Gaussian densities may play a crucial role in certain inequalities having some properties of affine invariance.

6.3.1. Motivation: Optimal Young inequality. To motivate the subsequent developments, we start from a very basic example of Gaussian inequality. Everybody knows the Young inequality for convolution:

$$\|f * g\|_{L^r(\mathbb{R}^n)} \le \|f\|_{L^p(\mathbb{R}^n)} \|g\|_{L^q(\mathbb{R}^n)}, \tag{6.10}$$

whenever $p, q, r \ge 1$ satisfy the compatibility condition

$$1 + \frac{1}{r} = \frac{1}{p} + \frac{1}{q}.$$

A proof of this inequality can be found in many elementary textbooks on real analysis.

Remark 6.14. The compatibility condition between the exponents is readily obtained by scaling homogeneity (replace f by $f(\cdot/\lambda)$ and note that the L^p norm scales like $\lambda^{n/p}$). For homogeneity arguments it is convenient to resort to formal reasoning just as a physicist would do: for instance, if ℓ is a length scale, then the left-hand side in (6.10) has homogeneity $[(\ell^n)^r \ell^n]^{1/r}$: the first ℓ^n is for the integration in the convolution operator, the second for

the integration inside the L^r norm, while $1/r$ is the power for the L^r norm. Hence, identification of the homogeneities yields $n + n/r = n/p + n/q$.

As we said, inequality (6.10) is quite well-known; however, it is not optimal! Whenever p, q are not 1 or ∞, then the constant 1 can be improved. The **optimal Young inequality** reads

$$\|f * g\|_{L^r(\mathbb{R}^n)} \leq \left(\frac{C_p C_q}{C_r}\right)^n \|f\|_{L^p(\mathbb{R}^n)} \|g\|_{L^q(\mathbb{R}^n)}, \tag{6.11}$$

where

$$C_p = \sqrt{\frac{p^{1/p}}{(p')^{1/p'}}}, \qquad \frac{1}{p} + \frac{1}{p'} = 1, \tag{6.12}$$

and the constant in front of the right-hand side of (6.11) is strictly less than 1 unless p or q is 1 or ∞.

Inequality (6.11) was proven independently by Beckner [**32**] and by Brascamp and Lieb [**56**] in the middle of the seventies. A short history of the subject can be found in [**32**]. This problem is also linked to the computation of the optimal constants in the **Hausdorff-Young inequality**, i.e. the operator norm of the Fourier transform from $L^p(\mathbb{R}^n)$ to $L^{p'}(\mathbb{R}^n)$, $p \geq 1$ (see [**190**] for an elementary reference).

Later, Lieb [**189**] proved that the only cases of equality are obtained for some particular Gaussian functions.

When one is interested in wild estimates from above, it is clear that the gain from (6.10) to (6.11) is negligible (for $p = q$, the gain on the constant is at most about 14%). But in many applications of mathematical physics, it is important to know the optimal constants in inequalities such as the convolution inequality, and to identify minimizers. Also the knowledge of optimal constants sometimes yields new inequalities as limit cases. For the optimal Young inequality, there is such an application, but it is rather anecdotical: as noticed by Lieb [**186**], inequality (6.11) implies as limit case (for p, q very close to 1) the **Shannon-Stam inequality**,

$$\mathcal{N}(f * g) \geq \mathcal{N}(f) + \mathcal{N}(g),$$

where $\mathcal{N}$ is Shannon's **entropy power functional**,

$$\mathcal{N}(f) = \frac{\exp\left(-\frac{2}{n}\int f \log f\right)}{2\pi e}, \tag{6.13}$$

well-known in information theory.

6.3.2. Brascamp-Lieb inequalities. There is a natural way to write the optimal Young inequality in a completely symmetric way. Using the duality between $L^r(\mathbb{R}^n)$ and $L^{r'}(\mathbb{R}^n)$ and the formula

$$\left|\int (f * g)\, h\right| = \left|\int f(x)g(x-y)h(y)\, dx\, dy\right|,$$

we can rewrite inequality (6.11) as

$$\begin{aligned}\left|\int f(x)g(x-y)h(y)\, dx\, dy\right| &\leq \left(\frac{C_p C_q}{C_r}\right)^n \|f\|_{L^p(\mathbb{R}^n)}\|g\|_{L^q(\mathbb{R}^n)}\|h\|_{L^{r'}(\mathbb{R}^n)} \\ &= (C_p C_q C_{r'})^n \|f\|_{L^p(\mathbb{R}^n)}\|g\|_{L^q(\mathbb{R}^n)}\|h\|_{L^{r'}(\mathbb{R}^n)}.\end{aligned}$$

Also note that, without loss of generality, we may restrict ourselves to nonnegative functions f, g, h.

In this form, the optimal Young inequality can be seen as a particular case of the **Brascamp-Lieb inequalities** [**56**]. We give below the more general form due to Lieb [**189**].

Theorem 6.15 (Generalized Brascamp-Lieb inequalities). *Let N and $(n_i)_{1\leq i\leq m}$ be integers ($n_i \leq N$), and let $(c_i)_{1\leq i\leq m}$ be positive real numbers such that*

$$\sum_{i=1}^{m} c_i n_i = N. \tag{6.14}$$

Let $B_i : \mathbb{R}^N \to \mathbb{R}^{n_i}$ be surjective linear mappings such that

$$\bigcap_i \ker B_i = \{0\}.$$

Whenever $f_1, \ldots, f_m$ are nonnegative L^1 functions, define

$$I(f_1, \ldots, f_m) = \int_{\mathbb{R}^N} \prod_i f_i^{c_i}(B_i x)\, dx. \tag{6.15}$$

Let $\overline{I}$ be the optimal constant in the inequality

$$I(f_1, \ldots, f_m) \leq \overline{I} \prod_i \left(\int_{\mathbb{R}^{n_i}} f_i\right)^{c_i}. \tag{6.16}$$

Then $\overline{I}$ can be computed by considering only centered Gaussian functions:

$$\overline{I} = \overline{I}_\gamma = \sup\left\{\frac{I(\gamma_1, \ldots, \gamma_m)}{\prod_i \left(\int_{\mathbb{R}^{n_i}} \gamma_i\right)^{c_i}};\quad \gamma_i \text{ centered Gaussian on } \mathbb{R}^{n_i}\right\}. \tag{6.17}$$

Exercise 6.16. Show that the compatibility condition (6.14) is necessary. Show that the theorem above reduces to the situation in (6.11) when $n_1 = n_2 = n$, $N = 2n$, $B_1(x, y) = x$, $B_2(x, y) = x - y$, $B_3(x, y) = y$, and find

the corresponding exponents c_i. Show that Hölder's inequality is another particular case (rather trivial) of Lieb's theorem.

Lieb's theorem results in a considerable reduction of the complexity of the problem in (6.16), which now becomes finite-dimensional. Using the identity $\int e^{-\langle Ax,x\rangle}\,dx = \pi^{n/2}/\sqrt{\det A}$, it is not very difficult to obtain the following expression (which is the best that one can do in general)

Lemma 6.17 (Brascamp-Lieb inequalities for Gaussians). *The constant $\overline{I}_\gamma$ in (6.17) is given by*

$$\overline{I}_\gamma = \frac{1}{\sqrt{D}},$$

$$D = \inf\left\{\frac{\det(\sum_i c_i B_i^* A_i B_i)}{\prod_i (\det A_i)^{c_i}};\quad A_i \in S_n^+(\mathbb{R}),\quad 1 \le i \le n\right\}. \tag{6.18}$$

6.3.3. Barthe's proof of Lieb's theorem. In this subsection, we present the very elegant proof by Barthe [**31**] of Lieb's theorem. This proof is based on two main tools: optimal transportation on one hand, and a certain duality between functional inequalities on the other hand. As a bonus, the proof yields a family of inequalities ("reverse Brascamp-Lieb inequalities", also called Barthe inequalities) which generalize the Prékopa-Leindler inequality and have interest on their own.

Theorem 6.18 (Reverse Brascamp-Lieb inequalities). *With the same notation as in Theorem 6.15, define the multilinear application*

$$J(g_1,\ldots,g_m) = \int_{\mathbb{R}^N}\left[\sup_{x=\sum c_i B_i^*(x_i)} \prod_i g_i^{c_i}(x_i)\right] dx. \tag{6.19}$$

Then, the optimal constant $\underline{J}$ in the inequality

$$J(g_1,\ldots,g_m) \ge \underline{J} \prod_i \left(\int_{\mathbb{R}^{n_i}} g_i\right)^{c_i} \tag{6.20}$$

can be computed by considering only centered Gaussian functions. Moreover,

$$\overline{I}\,\underline{J} = 1. \tag{6.21}$$

In this theorem, we used the notation B^* to denote the adjoint of a linear operator B. Inequality (6.20) is said to be dual to inequality (6.16), in a sense which is still to be made precise but is more or less apparent in the proof.

Example 6.19. The Prékopa-Leindler inequality is dual to Hölder's inequality (choose $n_1 = n_2 = N$, $c_1 = \lambda$, $c_2 = 1-\lambda$, $B_1 = B_2 = I_n$). We now understand why the Prékopa-Leindler inequality is "implied" by the "Prékopa-Leindler inequality for Gaussian functions"!

Proof of Theorems 6.15 and 6.18. Without loss of generality, the L^1-norms of $f_1, \ldots, f_m$ and $g_1, \ldots, g_m$ may be chosen all equal to 1. Thus

$$\overline{I} = \sup I(f_1, \ldots, f_m),$$
$$\underline{J} = \inf J(g_1, \ldots, g_m),$$

where in both cases the infimum/supremum is taken over all probability densities $f_1, \ldots, f_m$ or $g_1, \ldots, g_m$.

Next, we introduce

$$\overline{I}_\gamma = \sup I(\gamma_1, \ldots, \gamma_m),$$
$$\underline{J}_\gamma = \inf J(\gamma_1, \ldots, \gamma_m),$$

with infimum/supremum now taken over all centered Gaussian probability densities $\gamma_1, \ldots, \gamma_m$.

Of course $\overline{I} \geq \overline{I}_\gamma$, $\underline{J} \leq \underline{J}_\gamma$. Our goal is to show that $\overline{I} = \overline{I}_\gamma = 1/\sqrt{D}$ on one hand, $\underline{J} = \underline{J}_\gamma = \sqrt{D}$ on the other hand. The proof will be performed in two independent steps:

Step 1: Solve the Gaussian problem, i.e. prove that

$$\overline{I}_\gamma = \frac{1}{\sqrt{D}} = \frac{1}{\underline{J}_\gamma}. \tag{6.22}$$

This will be achieved by finite-dimensional duality, namely the usual Legendre duality for quadratic forms.

Step 2: Prove both theorems at the same time, after reduction to an inf/sup *problem.* In view of the result of Step 1, it is sufficient to prove that

$$\underline{J} \geq D\overline{I}. \tag{6.23}$$

Indeed, this will imply

$$\sqrt{D} = \underline{J}_\gamma \geq \underline{J} \geq D\overline{I} \geq D\overline{I}_\gamma = \sqrt{D},$$

and equality has to hold in all these inequalities, whence the conclusion. The great advantage of (6.23) is that it can be reformulated as an inf/sup problem:

$$D \sup I(f_1, \ldots, f_m) \leq \inf J(g_1, \ldots, g_m),$$

and this is of course equivalent to the following statement: for all probability densities $f_1, \ldots, f_m$, and $g_1, \ldots, g_m$, on $\mathbb{R}^n$, one has

$$D\, I(f_1, \ldots, f_m) \leq J(g_1, \ldots, g_m). \tag{6.24}$$

In the sequel, we first prove (6.22) and then (6.24), and this will conclude the proof.

Implementation of Step 1: Recall that $\overline{I}_\gamma = 1/\sqrt{D}$ from Lemma 6.17, so we just have to prove $\overline{I}_\gamma = 1/\underline{J}_\gamma$.

Letting A be a positive definite symmetric matrix, we denote by γ_A the centered Gaussian function

$$\gamma_A(x) = \sqrt{\frac{\det A}{\pi^n}} e^{-\langle x, Ax \rangle},$$

whose integral is normalized to 1. We use the standard convex duality for quadratic forms:

Definition 6.20 (Duality for quadratic forms). Let Q be a positive definite quadratic form on $\mathbb{R}^n$. We define

$$Q^\star(x) = \sup \left\{ |\langle x, y \rangle|^2; \quad Q(y) \leq 1 \right\}. \tag{6.25}$$

This duality is just the same as the Legendre duality, up to a factor 4 ($Q^\star$ in the sense of (6.25) is 4 times Q^* in the sense of Legendre duality). The advantage of this convention is to keep $|x|^2$ invariant, and hence to preserve the unit ball. We present the implementation of Step 1 as a series of exercises.

1. Check that if $Q(y) = \langle Ay, y \rangle$, then $Q^\star(x) = \langle A^{-1}x, x \rangle$.

2. Let $(c_i)_{1 \leq i \leq m}$ be nonnegative numbers and $B_i : \mathbb{R}^n \to \mathbb{R}^{n_i}$ be linear applications. Introduce

$$Q(y) = \sum_{i=1}^m c_i \langle B_i^* A_i B_i y, y \rangle.$$

Applying the Cauchy-Schwarz inequality to the quadratic form ϕ defined by $\phi(X_1, \ldots, X_m) = \sum_{i=1}^m \|X_i\|^2_{\mathbb{R}^{n_i}}$, check that

$$Q^\star(x) = \inf \left\{ \sum_{i=1}^m c_i \langle A_i^{-1} x_i, x_i \rangle; \quad x = \sum c_i B_i^* x_i, \ x_i \in \mathbb{R}^{n_i} \right\}. \tag{6.26}$$

This identity is actually related to the additive property of the (inverse) Legendre transform with respect to inf convolution (recall point 6) in subsection 2.1.3).

3. Deduce from (6.26) that

$$I(\gamma_{A_1}, \ldots, \gamma_{A_m}) \, J(\gamma_{A_1^{-1}}, \ldots, \gamma_{A_m^{-1}}) = 1$$

and conclude Step 1.

Implementation of Step 2: Let $f_1, \ldots, f_m$ and $g_1, \ldots, g_m$ be probability measures on $\mathbb{R}^{n_1}, \ldots, \mathbb{R}^{n_m}$ respectively. We have the following diagram:

$$\mathbb{R}^N \xrightarrow{B_i} \mathbb{R}^{n_i} \xrightarrow{B_i^*} \mathbb{R}^N.$$

Introduce optimal transportations $T_1, \dots, T_m$ such that $T_i \# f_i = g_i$ (here we again identify the probability measures $f_i\,dx_i$, $g_i\,dy_i$ with their densities). Thus

$$\mathbb{R}^N \xrightarrow{B_i} x_i \in \mathbb{R}^{n_i} \xrightarrow{T_i} y_i \in \mathbb{R}^{n_i} \xrightarrow{B_i^*} \mathbb{R}^N,$$

and we can write the Monge-Ampère equations

$$f_i = (g_i \circ T_i) \det(\nabla T_i), \tag{6.27}$$

where ∇T_i stands for the Jacobian matrix of T_i.

This induces a change of variables on $\mathbb{R}^N$,

$$\Theta(x) = \sum_{i=1}^m c_i B_i^* T_i(B_i x),$$

whose Jacobian matrix is given by

$$\nabla\Theta(x) = \sum_{i=1}^m c_i B_i^* \nabla T_i(B_i x) B_i.$$

Since each T_i is of the form $\nabla \varphi_i$, we have $\nabla T_i = D^2 \varphi_i$, and the change of variables Θ is monotone. (This can also be checked directly on the expression of Θ.) Under suitable regularity assumptions and after an approximation argument, one may use Caffarelli's regularity theory to see that Θ is a strictly monotone C^1 change of variables, well-defined from $\bigcap B_i^{-1}(\mathbb{R}^{n_i})$ into $\mathbb{R}^N$ (see [**31**]). By definition of the quantity D in (6.18),

$$\begin{aligned}\det[\nabla\Theta(x)] &= \det\left(\sum_{i=1}^m c_i B_i^* \nabla T_i(B_i x) B_i\right)\\ &\geq D \prod_{i=1}^m \left[\det \nabla T_i(B_i x)\right]^{c_i}.\end{aligned}$$

The proof of Step 2 is now straightforward:

$$\begin{aligned}J(g_1, \dots, g_m) &= \int_{\mathbb{R}^N} \sup\left\{\prod g_i(y_i)^{c_i};\quad y = \sum c_i B_i^* y_i\right\} dy\\ &= \int_{\mathbb{R}^N} \sup\left\{\prod g_i(y_i)^{c_i};\quad \sum c_i B_i^* y_i = \Theta(x)\right\} \det \nabla\Theta(x)\, dx\\ &\geq D \int_{\mathbb{R}^N} \sup\left\{\prod g_i(y_i)^{c_i};\quad \sum c_i B_i^* y_i = \Theta(x)\right\} \prod_i \left[\det \nabla T_i(B_i x)\right]^{c_i} dx.\end{aligned}$$

Of course, if each y_i is equal to $T_i(B_i x)$, then $\sum c_i B_i^* y_i = \Theta(x)$. Hence the last quantity is bigger than

$$D \int_{\mathbb{R}^N} \prod_i g_i(T_i(B_i x))^{c_i} \prod_i \left[\det \nabla T_i(B_i x)\right]^{c_i} dx$$

$$= D \int_{\mathbb{R}^N} \prod_i \left[g_i \circ T_i(B_i x) \det \nabla T_i(B_i x)\right]^{c_i} dx_i$$

$$= D \int_{\mathbb{R}^N} \prod_i f_i(B_i x)^{c_i} \, dx = D I(f_1, \ldots, f_m),$$

where the next-to-last equality is a consequence of the Monge-Ampère equation (6.27). □

6.4. Sobolev inequalities

Sobolev inequalities constitute another popular family of inequalities at the border between geometry and functional analysis. Whenever $n \geq 1$ is an integer and $p \geq 1$ is a real number, define the Sobolev space

$$W^{1,p}(\mathbb{R}^n) = \left\{ f \in L^p(\mathbb{R}^n); \quad \nabla f \in L^p(\mathbb{R}^n) \right\}.$$

When $p \in [1, n)$, define

$$p^\star = \frac{np}{n-p}. \tag{6.28}$$

Then the (critical) Sobolev embedding $W^{1,p}(\mathbb{R}^n) \subset L^{p^\star}(\mathbb{R}^n)$ asserts the existence of a finite constant $S_n(p) > 0$ such that

$$\forall f \in W^{1,p}(\mathbb{R}^n), \qquad \|f\|_{L^{p^\star}} \leq S_n(p) \left(\int_{\mathbb{R}^n} |\nabla f|^p \right)^{1/p}. \tag{6.29}$$

Without loss of generality, we assume that $S_n(p)$ is the smallest admissible (or optimal) constant in this inequality.

For the great majority of applications, it is not necessary to know more about the Sobolev embedding, except maybe explicit bounds on $S_n(p)$. However, in some circumstances one is interested in the exact value of the smallest admissible constant $S_n(p)$ in (6.29). There are usually two possible motivations for this: either for the computation of the ground state energy in some physical model, or because it provides some geometrical insights. For instance, in studies of isoperimetry on compact manifolds [**122, 123**] it is important to know that $S_n(p) \longrightarrow S_n(1)$ as $p \to 1^+$.

Of course, the value can be deduced from the identification of **extremal functions** in (6.29). The best constant $S_n(p)$ in (6.29) for $p > 1$ was first computed in the sixties, in unpublished work by Rodemich; then independently obtained by Aubin [**25**] and Talenti [**248**]. For $p = 1$ it has been known for a very long time that (6.29) with sharp constant is equivalent to the classical Euclidean isoperimetric inequality.

Below, we shall obtain the sharp constants in these Sobolev inequalities, as a simple application of the machinery developed in the previous chapters. The proof will follow [**101**]. With respect to other existing proofs, the

argument has the merit of being very elementary, and of applying to any norm (not necessary Euclidean) in $\mathbb{R}^n$. As a bonus, it exhibits an unexpected dual problem, just as Barthe's proof of Theorem 6.18.

For $1 < p < n$, we define the function h_p by

$$h_p(x) = \frac{1}{(\sigma_p + |x|^{p'})^{\frac{n-p}{p}}}, \tag{6.30}$$

where $p' = p/(p-1)$ is the dual exponent of p (not to be mistaken for $p^\star$), and σ_p is determined by the condition

$$\|h_p\|_{L^{p^\star}} = 1. \tag{6.31}$$

These functions will be the optimizers in the Sobolev inequality. Somewhat surprisingly, this property does not in fact depend on the choice of the norm. Note that h_p does not necessarily lie in L^p (which has no importance whatsoever).

Theorem 6.21 (Optimal Sobolev inequalities). *Let $p \in (1, n)$. Whenever $f, g \in L^{p^\star}(\mathbb{R}^n)$ are two functions satisfying $\|f\|_{L^{p^\star}} = \|g\|_{L^{p^\star}}$ and $\nabla f \in L^p(\mathbb{R}^n)$, then*

$$\frac{\displaystyle\int |g|^{p^\star(1-1/n)}}{\left(\displaystyle\int |y|^{p'}|g(y)|^{p^\star}\,dy\right)^{1/p'}} \leq \frac{p(n-1)}{n(n-p)}\,\|\nabla f\|_{L^p}, \tag{6.32}$$

and equality holds if $f = g = h_p$.

As immediate consequences we have

(i) the duality principle

$$\sup_{\|g\|_{L^{p^\star}}=1} \frac{\displaystyle\int |g|^{p^\star(1-1/n)}}{\left(\displaystyle\int |y|^{p'}|g(y)|^{p^\star}\,dy\right)^{1/p'}} = \frac{p(n-1)}{n(n-p)} \inf_{\|f\|_{L^{p^\star}}=1} \|\nabla f\|_{L^p} \tag{6.33}$$

with h_p extremal in both variational problems;

(ii) the sharp Sobolev inequality: if $f \neq 0$ lies in $L^{p^\star}(\mathbb{R}^n)$, then

$$\frac{\|\nabla f\|_{L^p}}{\|f\|_{L^{p^\star}}} \geq \|\nabla h_p\|_{L^p}; \tag{6.34}$$

(iii) the Sobolev embedding $W^{1,p}(\mathbb{R}^n) \subset L^{p^\star}(\mathbb{R}^n)$.

Proof of Theorem 6.21. It is clear that (6.32) implies both (i) and (ii), and that the latter is sharp. It is also clear that (iii) follows from (ii), because any function $f \in W^{1,p}$ can be approximated by functions f_k in $W^{1,p} \cap L^{p^\star}$ in such a way that $\|\nabla f_k\|_{L^p}$ converge to $\|\nabla f\|_{L^p}$. So we just have to prove the inequality (6.32) for arbitrary f and g. Thanks to the identity $|\nabla f| = |\nabla |f||$, we only need to consider the case when f is nonnegative. By a density argument, it is sufficient to consider the case when f and g are smooth and compactly supported. Also, by homogeneity, we only need to consider the case $\|f\|_{L^{p^\star}} = \|g\|_{L^{p^\star}} = 1$.

We introduce the two probability densities

$$F(x) = f^{p^\star}(x), \qquad G(y) = g^{p^\star}(y)$$

on $\mathbb{R}^n$. By Theorem 2.12 there exists a gradient of a convex function (uniquely determined almost everywhere on the support of f) such that

$$\nabla\varphi\#(F\,dx) = G\,dy.$$

Moreover, $\mathrm{Supp}(G) = \overline{\nabla\varphi(\mathrm{Supp}(F))}$.

Recall from Chapter 5 that the functional $F \longmapsto -\int F^{1-1/n}$ is displacement convex. This can be expressed by the above-tangent formulation of Section 5.2.6; more precisely, from Theorem 5.30 we find that

$$\int G^{1-\frac{1}{n}} \le \frac{1}{n}\int F^{1-\frac{1}{n}} \Delta_A \varphi. \tag{6.35}$$

Since G is compactly supported, it follows that $\nabla\varphi$ is bounded, and φ can be extended into a convex function on the whole of $\mathbb{R}^n$ (exercise). Then, since F is smooth and compactly supported, we can write

$$\frac{1}{n}\int F^{1-\frac{1}{n}}\,\Delta_A\varphi \le \frac{1}{n}\int F^{1-\frac{1}{n}}\,\Delta_{\mathcal{D}'}\varphi = -\frac{1}{n}\int \nabla(F^{1-\frac{1}{n}}) \cdot \nabla\varphi. \tag{6.36}$$

Returning to our original notation $F = f^{p^\star}$ and $G = g^{p^\star}$, we have just shown, combining (6.35) and (6.36), that

$$\int g^{\frac{p(n-1)}{n-p}} \le -\frac{p(n-1)}{n(n-p)}\int f^{\frac{n(p-1)}{n-p}}\nabla f \cdot \nabla\varphi = -\frac{p(n-1)}{n(n-p)}\int f^{p^\star/p'}\nabla f \cdot \nabla\varphi. \tag{6.37}$$

By **Hölder's inequality** (in its vector-valued version),

$$-\int f^{p^\star/p'}\nabla f\cdot\nabla\varphi \le \|\nabla f\|_{L^p}\left(\int f^{p^\star}|\nabla\varphi|^{p'}\right)^{1/p'}. \tag{6.38}$$

But, by the definition of push-forward, $\displaystyle\int f^{p^\star}|\nabla\varphi|^{p'} = \int |y|^{p'} g^{p^\star}(y)\,dy$. Therefore the combination of (6.37) and (6.38) concludes the proof of inequality (6.32).

Let us now choose $f = g = h_p$, and check that equality holds at all the steps of the proof, and therefore in (6.32). Of course this function is not compactly supported, but in this particular case the Brenier map reduces to the identity map $\nabla\varphi(x) = x$, and all the steps can be checked explicitly. Indeed, $\nabla\varphi(x) = x$ leads to an equality in (6.35) and in (6.36) (via integration by parts). Then one can also check that there is equality in (6.38). This ends the proof of Theorem 6.21. $\square$

Remarks 6.22. (i) The choice $f = g = h_p$ is not mysterious: it can be guessed by looking at equality cases in Hölder's inequality. In fact, equality in (6.38) implies $\|\nabla f(x)\|^p = k f^{p^\star}(x)\|\nabla\varphi(x)\|^{p'}$ for almost all $x \in \mathbb{R}^n$. If we now assume $\nabla\varphi(x) = x$, and look for radially symmetric minimizers, we arrive at h_p.

(ii) In the present case, inequality (6.35) can be proven in a more direct way without invoking Theorem 5.30: using the definition of push-forward and the Monge-Ampère equation (4.10), one can write

$$\begin{aligned}\int G(y)\, G(y)^{-1/n}\, dy &= \int F(x)\, G(\nabla\varphi(x))^{-1/n}\, dx \\ &= \int F(x) F(x)^{-1/n} [\det D_A^2\varphi(x)]^{1/n}\, dx;\end{aligned}$$

then (6.35) follows by the inequality $(\det D_A^2\varphi)^{1/n} \leq (\Delta_A\varphi)/n$, which is another instance of the arithmetic-geometric inequality (Lemma 5.23).

(iii) The very same proof works, mutatis mutandis, for arbitrary norms on $\mathbb{R}^n$. Letting $\|\cdot\|$ be a norm on $\mathbb{R}^n$, we define its dual norm by

$$\|X\|_* = \sup_{\|Y\|\leq 1} X \cdot Y.$$

Then the Sobolev norm is defined by $\left(\int \|\nabla f\|_*^p\right)^{1/p}$, and the minimizers by $(\sigma_p + \|x\|^{p'})^{-\frac{n-p}{p}}$.

(iv) It is possible to establish (6.37) directly, even when F and G are neither smooth, nor compactly supported, assuming only that $f \in W^{1,p}(\mathbb{R}^n)$ and $g \in L^{p^\star}(\mathbb{R}^n)$ are two nonnegative functions such that $\|f\|_{L^{p^\star}} = \|g\|_{L^{p^\star}} = 1$ and $\int g^{p^\star}(y)\, |y|^{p'}\, dy < +\infty$. A proof is given in [**101**].

(v) By tracing cases of equality in the inequalities used above, one can prove

Theorem 6.23 (Cases of equality in the Sobolev inequality). *A function $f \in L^{p^\star}(\mathbb{R}^n)$ is optimal in the Sobolev inequality (6.34) if and only if there exist $C \in \mathbb{R}$, $\lambda \neq 0$ and $x_0 \in \mathbb{R}^n$ such that*

$$f(x) = C\, h_p\big(\lambda(x - x_0)\big). \tag{6.39}$$

Again, the theorem applies to general norms in $\mathbb{R}^n$. The proof based on the strategy above is rather technical, see [**101**]; it is however somewhat simpler than the "classical" proof, based on rearrangement inequality and reduction to a one-dimensional problem (the inequality $\|\nabla f^*\|_{L^p} \leq \|\nabla f\|_{L^p}$, where f^* is the monotone radially symmetric rearrangement of f, is known as the **Pólya-Szegö principle**; see Lieb [**188**] for a short proof based on the Riesz rearrangement inequality in the case $p = 2$).

In [**101**] it is also shown how to obtain by the same method the optimal Gagliardo-Nirenberg inequalities proven by Dolbeault and Del Pino [**113**], which will be alluded to again in Section 9.6. Generally speaking, Gagliardo-Nirenberg inequalities take the form

$$\|f\|_{L^r} \leq G_n(p,r,s)\|\nabla f\|_{L^p}^{\theta}\|f\|_{L^s}^{1-\theta}, \tag{6.40}$$

where $n \geq 2$, $p \in (1,n)$, $r \leq p^\star$, $s < r$, and $\theta = \theta(n,p,r,s) \in (0,1)$ is determined by scaling invariance. They can actually be deduced from (6.29) by means of the Hölder inequality (exercise). Apart from some isolated cases, optimal constants are known only for the following subfamily:

$$\begin{cases} p(s-1) = r(p-1) & \text{when } r,s > p, \\ p(r-1) = s(p-1) & \text{when } r,s < p. \end{cases}$$

Exercise 6.24 (Isoperimetric inequality again). Prove the following version of the isoperimetric inequality. Let

$$h_1(x) = \frac{1_B(x)}{|B|},$$

where B stands for the Euclidean n-dimensional unit ball. Let $f \neq 0$ be a smooth compactly supported function. Then

$$\frac{\|\nabla f\|_{L^1}}{\|f\|_{L^{n/(n-1)}}} \geq n|B|^{\frac{1}{n}}. \tag{6.41}$$

Hints: By the same pattern of proof as for Theorem 6.21, show that, when $\|f\|_{L^{n/(n-1)}} = 1$, then there exists $\nabla\varphi : \mathbb{R}^n \to B$ such that

$$|B|^{1/n} \leq -\frac{1}{n}\int \nabla f \cdot \nabla\varphi \leq \|\nabla f\|_{L^1}. \tag{6.42}$$

The scheme of this proof is due to Gromov [**208**, Appendix]. Again, it works for any norm on $\mathbb{R}^n$.

Remark 6.25. Inequality (6.41) extends to functions with bounded variation (i.e., whose distributional gradient is a measure), with equality if $f = h_1$. By approximating h_1 by a well-chosen family of continuous functions, one can deduce from inequality (6.41) the isoperimetric inequality (6.4) again. Since this inequality is sharp, (6.42) must also be sharp.

Chapter 7

The Metric Side of Optimal Transportation

So far, we have only been concerned with the first basic question about the Monge-Kantorovich problem: prove existence of an optimal transportation and characterize it. In this chapter we shall briefly study the next question: *what information does it give on μ and ν to know the value $\mathcal{T}_c(\mu, \nu)$ of the optimal transportation cost?*

This will lead us to the study of what we shall call Monge-Kantorovich distances. The study of their properties is somewhat simpler than the problem of characterizing the optimal transportation, and will require much less structure. This is why, as in Chapter 1, we shall work under quite general assumptions on the space X, which will not be assumed to be Euclidean or even to have a differentiable structure. Most of the time, we shall only require X to be a Polish space, i.e. a separable complete metric space. This choice of generality is not academic; it is motivated by several concrete applications in which it is of interest to use Monge-Kantorovich distances on spaces like $C([0,1];\mathbb{R}^d)$ or $P(\mathbb{R}^d)$.

From the beginning we display two important warnings.

First, we shall make no attempt in this chapter to cover the considerable literature related to Monge-Kantorovich distances, and we refer to Rachev [**226**], Rachev and Rüschendorf [**225**] for much more. For other applications as well as historical remarks, the reader may also consult Dudley [**124, 127**], inspired by problems in mathematical statistics. Here we

shall only establish the basic properties which we will need later. On the other hand, we shall establish them with a lot of generality, keeping in mind that they are useful in many different contexts.

Secondly, the terminology which is associated with these distances varies a lot from one mathematical community to another, and we do not claim that our denominations are really accurate (in fact, we are pretty sure that they are not). Some Monge-Kantorovich distances may be encountered under the names of Tanaka distance, Kantorovich-Rubinstein distance, Wasserstein distance, minimal metrics, etc. In particular, we shall speak of *quadratic Wasserstein distance* when the cost is the square of the distance, and *Kantorovich-Rubinstein distance* when the cost is the distance. (The latter terminology is certainly historically accurate, while the former is definitely not, but has gained widespread popularity recently.)

Reminders 7.1. Before going on, it is preferable to have ideas clear about measures in a Polish space. We already made all the necessary reminders in Chapter 1; but let us recall them very briefly. By definition a Polish space is a separable, complete, metric space (endowed with the topology induced by the metric). Whenever X is a Polish space, we denote by $P(X)$ the space of Borel probability measures, i.e. those which are defined on the Borel σ-algebra of X. Here are a few basic facts about the topology of $P(X)$, as can be found in [**43, 127**] for instance. First, any Borel probability measure on X is regular and has σ-compact support. By definition, μ_k converges weakly to μ if for all $\varphi \in C_b(X)$ (i.e., φ is bounded and continuous), $\int \varphi \, d\mu_k$ converges to $\int \varphi \, d\mu$ as $k \to \infty$. This defines a separable, Hausdorff topology on $P(X)$, called the *weak topology.* Included in this statement is the fact that $\mu = \nu$ if and only if for all $\varphi \in C_b(X)$, $\int \varphi \, d\mu = \int \varphi \, d\nu$. Moreover, Prokhorov's theorem ensures that a subset S of $P(X)$ is relatively weakly compact if and only if it is tight, i.e. for all $\varepsilon > 0$ there is a compact subset K_ε of X such that for all $\mu \in S$, $\mu[K_\varepsilon^c] \leq \varepsilon$.

If, in addition to being Polish, X is *locally compact* (each point admits a compact neighborhood, as in $\mathbb{R}^n$), then Riesz' theorem identifies the space $M(X)$ of measures, normed by total variation, with the dual of the space $C_0(X)$ of continuous functions going to 0 at infinity. Then one can introduce the "weak-$*$ topology" on $P(X)$, or more generally on $M(X)$; it is defined just as the weak topology above, but with the test function space $C_b(X)$ replaced by $C_0(X)$. Since we do not assume local compactness, we will avoid this terminology.

Remark 7.2. In the locally compact case, at the level of probability measures, weak and weak-$*$ *convergences* (not weak and weak-$*$ topologies!!) are in fact *equivalent* — discrepancies occur only when one is interested in the larger space $M(X)$. For instance, a sequence of probability measures may

converge in the weak-$*$ sense (but not in the weak sense) to a nonnegative measure which is not a probability measure.

7.1. Monge-Kantorovich distances

7.1.1. Definitions. Let X be a Polish space endowed with a distance d. Let $p \geq 0$ be a nonnegative real number. We shall consider the cost function $c(x,y) = d(x,y)^p$, with the convention that $d(x,y)^0 = 1_{x \neq y}$. We shall use the abbreviation $\mathcal{T}_p(\mu,\nu) = \mathcal{T}_{d^p}(\mu,\nu)$ for the associated optimal transportation cost between two probability measures μ and ν on X. We shall denote by $P_p(X)$ the set of probability measures with finite moments of order p, i.e. those measures μ such that for some (and thus any) $x_0 \in X$,

$$\int d(x_0,x)^p \, d\mu(x) < +\infty.$$

Of course, if d is bounded, then $P_p(X)$ coincides with the set $P(X)$ of all probability measures on X.

Theorem 7.3 (**Wasserstein distances**). *(i) For all $p \in [1,\infty)$, $W_p = \mathcal{T}_p^{1/p}$ defines a metric on $P_p(X)$.*

(ii) For all $p \in [0,1)$, $W_p = \mathcal{T}_p$ defines a metric on $P_p(X)$.

Remark 7.4. If d is bounded, then, as a corollary of the preceding theorem, W_p defines a metric on $P(X)$. To exploit this remark, if d is unbounded, it is sometimes convenient to replace it by the bounded metric $\widetilde{d} = \inf(d,1)$, which induces the same topology as d.

A few words about terminology. We shall call W_p the Monge-Kantorovich distance of order p, or Monge-Kantorovich distance with exponent p. The Monge-Kantorovich distance with exponent 2, $W_2 = \mathcal{T}_2^{1/2}$, will be called the **quadratic Wasserstein distance**. The Monge-Kantorovich distance with exponent 1, $W_1 = \mathcal{T}_1$, will be called the **Kantorovich-Rubinstein distance**. As for the Monge-Kantorovich distance of order 0, $W_0 = \mathcal{T}_0$, it is just half of the total variation norm, by formula (13).

Remarks 7.5. (i) The Kantorovich-Rubinstein distance can also be defined in an alternative way by the Kantorovich-Rubinstein duality formula,

$$W_1(\mu,\nu) = \sup_{\|\varphi\|_{\mathrm{Lip}} \leq 1} \int_X \varphi \, d(\mu - \nu). \tag{7.1}$$

Note that if d is bounded by some positive real number M, and φ is 1-Lipschitz with respect to d, then $\sup \varphi - \inf \varphi \leq M$, and therefore, without loss of generality, one can require that $0 \leq \varphi \leq M$ in (7.1). Moreover, the supremum can be restricted to bounded Lipschitz functions. The distance

W_1 is sometimes called the **bounded Lipschitz distance**, although this is not the most standard terminology, recall Remark 1.15 (iii).

(ii) When μ is a probability measure on a Banach space X, and a is any element of X, then

$$W_2(\mu, \delta_a)^2 = \int_X \|x - a\|^2 \, d\mu(x).$$

In particular, the *mean* of μ, which is defined as $m = \int x \, d\mu(x)$, is the unique solution of the minimization problem

$$\inf_{a \in X} W_2(\mu, \delta_a),$$

and the corresponding cost $\mathcal{T}_2$ is just the *variance* of μ.

Proof of Theorem 7.3. We shall only prove statement (i), since statement (ii) is a particular case of (i) if $0 < p < 1$ (replace d by the topologically equivalent distance d^p) and can be seen as a consequence of (13) if $p = 0$. Thus we assume $p \geq 1$.

Of course W_p is symmetric, nonnegative, and we leave it as an exercise to check that it is finite on $P_p(X)$.

It is clear that $W_p(\mu, \mu) = 0$. Conversely, let μ, ν be two probability measures such that $W_p(\mu, \nu) = 0$; we shall prove that $\mu = \nu$. Let π be an optimal transportation plan (recall from Chapter 1 that there exists at least one); it is clear that $d\pi(x, y)$ is supported on the diagonal $(y = x)$. Thus, for all $\varphi \in C_b(X)$, $\int \varphi \, d\mu = \int \varphi(x) \, d\pi(x, y) = \int \varphi(y) \, d\pi(x, y) = \int \varphi \, d\nu$, which implies $\mu = \nu$, as announced.

All that remains to check is the triangle inequality. We shall obtain it as a consequence of an important lemma which enables us to "glue together" two transference plans having a common marginal.

Lemma 7.6 (Gluing lemma). *Let μ_1, μ_2, μ_3 be three probability measures, supported in Polish spaces X_1, X_2, X_3 respectively, and let $\pi_{12} \in \Pi(\mu_1, \mu_2)$, $\pi_{23} \in \Pi(\mu_2, \mu_3)$ be two transference plans. Then there exists a probability measure $\pi \in P(X_1 \times X_2 \times X_3)$ with marginals π_{12} on $X_1 \times X_2$ and π_{23} on $X_2 \times X_3$.*

With this lemma at hand, let us conclude the proof of the triangle inequality. We consider μ_1, μ_2, μ_3 in $P_p(X)$, and optimal transference plans π_{12} between μ_1 and μ_2, and π_{23} between μ_2 and μ_3 (in fact, at this point it is not really necessary to invoke the existence of optimal transference plans, as the reader can check). We choose X_i to be the support of μ_i. Let π be as in the gluing lemma, and let π_{13} be the marginal of π on $X_1 \times X_3$. Clearly, $\pi_{13} \in \Pi(\mu_1, \mu_3)$. Using succesively the definition of $\mathcal{T}_p$, the marginal property, the triangle inequality and Minkowski's inequality for L^p functions, we

obtain

$$W_p(\mu_1,\mu_3) \le \left(\int_{X_1\times X_3} d(x_1,x_3)^p\, d\pi_{13}(x_1,x_3)\right)^{1/p}$$
$$= \left(\int_{X_1\times X_2\times X_3} d(x_1,x_3)^p\, d\pi(x_1,x_2,x_3)\right)^{1/p}$$
$$\le \left(\int_{X_1\times X_2\times X_3} [d(x_1,x_2)+d(x_2,x_3)]^p\, d\pi(x_1,x_2,x_3)\right)^{1/p}$$

$$\le \left(\int_{X_1\times X_2\times X_3} d(x_1,x_2)^p\, d\pi(x_1,x_2,x_3)\right)^{1/p}$$
$$+ \left(\int_{X_1\times X_2\times X_3} d(x_2,x_3)^p\, d\pi(x_1,x_2,x_3)\right)^{1/p}$$
$$= \left(\int_{X_1\times X_2} d(x_1,x_2)^p\, d\pi_{12}(x_1,x_2)\right)^{1/p} + \left(\int_{X_2\times X_3} d(x_2,x_3)^p\, d\pi_{23}(x_2,x_3)\right)^{1/p}$$
$$= W_p(\mu_1,\mu_2) + W_p(\mu_2,\mu_3).$$

□

Exercise 7.7. In the case $p = 1$, prove the triangle inequality from the Kantorovich-Rubinstein dual representation.

We now give the proof of Lemma 7.6. This lemma was apparently first proven by Vorob'ev for finite sets, later generalized by Berkes and Philipp, then Shortt (see Dudley [**126**, p. 20] for comments about the history and the generality of these results, which are often considered folklore).

Proof of Lemma 7.6. This is the occasion to introduce the concept of **disintegration of measure** (see [**157**] for accurate and up-to-date references). When X and Y are Polish spaces, the disintegration of measure theorem allows one to write any probability measure on $X \times Y$ as an average of probability measures on $\{x\} \times Y$, for $x \in X$. In particular, if π is a probability measure on $X \times Y$, with marginal μ on X, then there exists a measurable application $x \mapsto \pi_x$, from X into $P(Y)$, uniquely determined $d\mu(x)$-almost everywhere, such that

$$\pi = \int_X (\delta_x \otimes \pi_x)\, d\mu(x). \tag{7.2}$$

Formula (7.2) means of course that for all $u \in C_b(X \times Y)$,

$$\int_{X\times Y} u(x,y)\, d\pi(x,y) = \int_X \left[\int_Y u(x,y)\, d\pi_x(y)\right] d\mu(x),$$

or that for any measurable set $A \subset X \times Y$,

$$\pi[A] = \int_X \pi_x[A_x]\, d\mu(x),$$

where

$$A_x = \{y \in Y;\ (x,y) \in A\}.$$

Now, consider π_{12} and π_{23} as in the statement of Lemma 7.6, and disintegrate both measures with respect to their common marginal μ_2. Thus there exist measurable mappings $\pi_{12;2}$ and $\pi_{23;2}$, from X_2 into $P(X_1)$, $P(X_3)$ respectively, such that

$$\pi_{12} = \int_{X_2} \pi_{12;2} \otimes \delta_{x_2}\, d\mu_2(x_2),$$

$$\pi_{23} = \int_{X_2} \delta_{x_2} \otimes \pi_{23;2}\, d\mu_2(x_2).$$

We construct $\pi \in P(X_1 \times X_2 \times X_3)$ by setting

$$\pi = \int_{X_2} \left(\pi_{12;2} \otimes \delta_{x_2} \otimes \pi_{23;2}\right) d\mu_2(x_2).$$

We leave it to the reader to check that π satisfies all the desired properties. □

Exercise 7.8. In the statement of disintegration of measure, construct π_x explicitly when all the measures in consideration are absolutely continuous with respect to a fixed reference measure.

Exercise 7.9. Assuming that all X_i's are compact, give an alternative proof of Lemma 7.6 by means of the Hahn-Banach extension theorem.

Hint: Introduce the vector space generated by functions φ_{12} of x_1, x_2 and φ_{23} of x_2, x_3, and consider the linear functional Θ defined (well, you have to show that it is well-defined) by

$$\Theta[\varphi_{12} + \varphi_{23}] = \int_{X_1 \times X_2} \varphi_{12}\, d\pi_{12} + \int_{X_2 \times X_3} \varphi_{23}\, d\pi_{23}.$$

7.1.2. Ordering. A fundamental property of the distances W_p is that they are ordered. Indeed, from Hölder's inequality one deduces at once that

$$p_1 \geq p_2 \geq 1 \implies W_{p_1} \geq W_{p_2}. \tag{7.3}$$

It is not in general possible to compare W_{p_1} and W_{p_2} in the other way unless d is *bounded*. If this condition is fulfilled, it is an easy exercise to establish the interpolation bounds

$$p_1 \geq p_2 \geq 1 \Longrightarrow W_{p_1} \leq W_{p_2}^{\frac{p_2}{p_1}} \operatorname{diam}(X)^{1-\frac{p_2}{p_1}}, \tag{7.4}$$

where $\operatorname{diam}(X) = \sup\{d(x,y);\ x,y \in X\}$ is the diameter of X. Thus, in this situation, all distances W_p ($p \geq 1$) define the same topology on $P(X)$.

7.1.3. Bounds. The following bounds, adapted from [**225**, Lemma 10.2.3], are often too strong, but sometimes quite useful.

Proposition 7.10 (Weighted total variation control on Wasserstein distances). *Let μ and ν be two probability measures on a Polish space X and let d be a distance on X. Then, for any $p \geq 0$ and any $x_0 \in X$,*

$$\begin{aligned}\mathcal{T}_p(\mu,\nu) &\leq \max(1, 2^{p-1}) \int d(x_0,x)^p \, d|\mu-\nu|(x) \\ &= \max(1,2^{p-1}) \big\| d(x_0,\cdot)^p(\mu-\nu)\big\|_{TV}.\end{aligned}$$

Proof. Let π be the transference plan obtained by keeping fixed all the mass shared by μ and ν, and distributing the rest uniformly: this is

$$\pi = (\mathrm{Id} \times \mathrm{Id})\#(\mu\wedge\nu) + \frac{1}{a}\,(\mu-\nu)_+ \otimes (\mu-\nu)_-,$$

where $\mu\wedge\nu = \mu - (\mu-\nu)_+$ and $a = (\mu-\nu)_-[X] = (\mu-\nu)_+[X]$. A more sloppy but probably more readable way to write π is

$$d\pi(x,y) = d(\mu\wedge\nu)(x)\delta_{y=x} + \frac{1}{a} d(\mu-\nu)_+(x)\, d(\mu-\nu)_-(y).$$

By using the definition of $\mathcal{T}_p$, the definition of π, the triangle inequality for d, the elementary inequality $(A+B)^p \leq \max(1,2^{p-1})(A^p+B^p)$, and the definition of a, we find that

$$\begin{aligned}\mathcal{T}_p(\mu,\nu) &\leq \int d(x,y)^p\, d\pi(x,y) \\ &= \frac{1}{a}\int d(x,y)^p \, d(\mu-\nu)_+(x)\, d(\mu-\nu)_-(y) \\ &\leq \frac{\max(1,2^{p-1})}{a} \int \big[d(x,x_0)^p + d(x_0,y)^p\big]\, d(\mu-\nu)_+(x)\, d(\mu-\nu)_-(y) \\ &\leq \max(1,2^{p-1}) \left[\int d(x,x_0)^p\, d(\mu-\nu)_+(x) + \int d(x_0,y)^p \, d(\mu-\nu)_-(y)\right] \\ &= \max(1,2^{p-1}) \int d(x,x_0)^p\, d\big[(\mu-\nu)_+ + (\mu-\nu)_-\big](x) \\ &= \max(1,2^{p-1}) \int d(x,x_0)^p\, d|\mu-\nu|(x).\end{aligned}$$

□

7.2. Topological properties

In this section we consider the topological properties of the Monge-Kantorovich distances W_p. The space X is still assumed to be a Polish space, endowed with a distance d.

A first hint of the topological properties to be expected from W_p is given by the trivial remark that for all $x, y \in X$,

$$W_p(\delta_x, \delta_y) = d(x,y)^{\inf(1,p)}.$$

In particular, for $p > 0$, $W_p(\delta_x, \delta_y)$ converges to 0 when $x \to y$.

A second hint is given by the property that W_p is not very sensitive to wild oscillations. To illustrate this in a concrete way, we suggest the following:

Exercise 7.11 (The Wasserstein distance is insensitive to oscillations). For $k \geq 1$, consider $f_k(x) = 1 + \sin(2\pi k x)$ on the line segment $[0,1]$. Let $d\mu_k(x) = f_k(x)\,dx$, and let $\mathcal{L}$ be the Lebesgue measure on $[0,1]$. Let $p \geq 1$; show that $W_p(\mu_k, \mathcal{L}) \leq C_p k^{-1}$ for some constant C_p depending only on p.

After these preliminary remarks, the following theorem should appear natural:

Theorem 7.12 (Wasserstein distances metrize weak convergence). *Let $p \in (0,\infty)$, let $(\mu_k)_{k\in\mathbb{N}}$ be a sequence of probability measures in $P_p(X)$, and let $\mu \in P(X)$. Then, the following four statements are equivalent:*

(i) $W_p(\mu_k, \mu) \xrightarrow[k\to\infty]{} 0.$

(ii) $\mu_k \xrightarrow[k\to\infty]{} \mu$ *in the weak sense, and* (μ_k) *satisfies the following* **"tightness"** *condition: for some (and thus any)* $x_0 \in X$, *we have*

$$\lim_{R\to\infty} \limsup_{k\to\infty} \int_{d(x_0,x)\geq R} d(x_0,x)^p \, d\mu_k(x) = 0. \tag{7.5}$$

(iii) $\mu_k \xrightarrow[k\to\infty]{} \mu$ *in the weak sense,* μ *has a finite moment of order* p, *and there is convergence of that moment of order* p *as* $k \to \infty$*; that is, for some (and thus any)* $x_0 \in X$,

$$\int d(x_0,x)^p \, d\mu_k(x) \xrightarrow[k\to\infty]{} \int d(x_0,x)^p \, d\mu(x) < +\infty. \tag{7.6}$$

(iv) Whenever a continuous function φ *on* X *satisfies the growth condition* $|\varphi(x)| \leq C[1 + d(x_0,x)^p]$ *for some* $x_0 \in X$, $C \in \mathbb{R}$, *then* $\int |\varphi(x)|\,d\mu(x) < +\infty$, *and*

$$\int \varphi \, d\mu_k \xrightarrow[k\to\infty]{} \int \varphi \, d\mu.$$

Remarks 7.13. (i) The statements "and thus any" in points (ii) and (iii) mean the following: if condition (ii) or (iii) is satisfied for some x_0, then it is satisfied for any x_0. This is a consequence of the triangle inequality; details are left as an exercise for the reader.

(ii) As a consequence of this theorem, W_p metrizes the weak topology of measures on any subset of $P_p(X)$ which satisfies a uniform "tightness" condition, as described in (ii). Although we shall not prove it here, this metric turns out to be complete (see e.g. [**267**, Chapter 6]).

(iii) In particular, if d is bounded, then W_p metrizes weak convergence on the whole of $P(X)$. If d is unbounded, we may also replace it by a topologically equivalent bounded distance; it follows that $P(X)$, endowed with the weak topology, is a metric space (there are however simpler ways to see this). Of course this is consistent with the fact, which we admitted, that the weak topology on $P(X)$ is completely determined by the definition of convergent *sequences* of probability measures. In fact, $P(X)$ turns out to be a Polish space itself — which leads to the useful possibility of considering Monge-Kantorovich distances on $P(P(X))$.

(iv) It follows from the triangle inequality that $W_p(\mu_k, \nu_k) \longrightarrow W_p(\mu, \nu)$ as soon as $W_p(\mu_k, \mu) \longrightarrow 0$ and $W_p(\nu_k, \nu) \longrightarrow 0$. (This can also be seen as a consequence of the Kantorovich duality.)

(v) In the case $p = 1$, the right-hand side of (7.1) can obviously be extended to arbitrary Radon measures on X, and it is easy to check that it yields a distance, and even a norm, on the set $M(X)$ of all Radon measures ("bounded Lipschitz norm"). However, it *does not* metrize weak convergence on $M(X)$, unless X is a finite space. In fact, if E is any Banach space, then the weak-$*$ topology on E^* cannot be metrized unless E is finite-dimensional.

(vi) The conclusion of the theorem does not in general hold true for the limit case $p = 0$ — not even if the distance d is replaced by d^0. Indeed, $\mu_k \to \mu$ weakly in (X, d^0) means that for all bounded measurable functions φ, $\int \varphi \, d\mu_k$ converges to $\int \varphi \, d\mu$ as $k \to \infty$; a particular case of this is the *weak* convergence of $d\mu_k/d\mu$ to 1 in $L^1(d\mu)$. On the other hand, $W_0(\mu_k, \mu) \to 0$ means convergence of μ_k to μ in the total variation norm, a particular case of which is the *strong* convergence of $d\mu_k/d\mu$ to 1 in $L^1(d\mu)$.

(vii) Wasserstein metrics are definitely not the only interesting way to metrize a space of probability measures. Many other metrics have been used in probability theory (Lévy metric, ideal metrics, etc.). Rachev's book [**226**] gives a detailed account on the subject, both theory and applications. In this reference the author has compiled a list of seventy-six metrics for probability measures! One may also consult the book by Kalashnikov and Rachev [**174**], or the one by Zolotarev [**273**], in which a chapter is devoted to probability

metrics. Finally, one can find in Hanin [**165**] (and the references there included) a discussion of the theory of *Kantorovich norms*, which are variants of the Monge-Kantorovich norm (1.20), defined on the whole space of Radon measures.

Proof of Theorem 7.12. 1. Let the assumptions of the theorem be satisfied. Without loss of generality, we assume $p \geq 1$. We first check that (ii), (iii) and (iv) are equivalent. Obviously (iv) implies (iii). Assume now that (ii) is satisfied for some $x_0 \in X$. Pick an arbitrary function φ satisfying the growth condition in (iv), and write, for any $R > 1$,

$$\varphi = \varphi_R + \psi_R,$$

where $\varphi_R(x) = \inf(\varphi(x), C(1+R^p))$ and $\psi_R(x) = \varphi(x) - \varphi_R(x)$ is pointwise bounded by $Cd(x_0,x)^p 1_{d(x_0,x)\geq R}$. Then,

$$\left|\int \varphi\, d\mu_k - \int \varphi\, d\mu\right| \leq \left|\int \varphi_R\, d(\mu_k - \mu)\right| + C\int_{d(x_0,x)\geq R} d(x_0,x)^p\, d\mu_k(x) \\ + C\int_{d(x_0,x)\geq R} d(x_0,x)^p\, d\mu(x).$$

So,

$$\limsup_{k\to\infty}\left|\int \varphi\, d\mu_k - \int \varphi\, d\mu\right| \leq C\limsup_{k\to\infty}\int_{d(x_0,x)\geq R} d(x_0,x)^p [d\mu_k + d\mu](x).$$

We then let $R \to \infty$ and notice that the right-hand side goes to 0 in the limit. This shows that (ii) implies (iv).

Next, let us check that (iii) implies (ii). With the notation $a \wedge b = \inf(a,b)$, we have

$$\int [d(x_0,x)\wedge R]^p\, d\mu_k(x) \xrightarrow[k\to\infty]{} \int [d(x_0,x)\wedge R]^p\, d\mu(x);$$

on the other hand, by the monotone convergence theorem,

$$\lim_{R\to\infty}\int [d(x_0,x)\wedge R]^p\, d\mu(x) = \int d(x_0,x)^p\, d\mu(x);$$

finally, by assumption (iii),

$$\int d(x_0,x)^p\, d\mu_k(x) \xrightarrow[k\to\infty]{} \int d(x_0,x)^p\, d\mu(x).$$

We conclude that

$$\lim_{R\to\infty}\lim_{k\to\infty}\int \left[d(x_0,x)^p - (d(x_0,x)\wedge R)^p\right] d\mu_k(x) = 0.$$

When $d(x_0, x) \geq 2R$, obviously $d(x_0, x)^p - R^p \geq (1-2^{-p}) d(x_0, x)^p$. It follows that

$$\lim_{R\to\infty} \limsup_{k\to\infty} \int_{d(x_0,x)\geq 2R} d(x_0, x)^p \, d\mu_k(x) = 0,$$

which is (ii).

2. To prove the theorem it only remains to check that (i) is equivalent to (iii). Notice that the weak convergence of μ_k to μ implies

$$\begin{aligned}\int d(x_0, x)^p \, d\mu(x) &= \lim_{R\to\infty} \lim_{k\to\infty} \int [d(x_0, x) \wedge R]^p \, d\mu_k(x) \\ &\leq \liminf_{k\to\infty} \int d(x_0, x)^p \, d\mu_k(x),\end{aligned}$$

and therefore the condition of convergence of the moment of order p in (iii) is equivalent to

$$\limsup_{k\to\infty} \int d(x_0, x)^p \, d\mu_k(x) \leq \int d(x_0, x)^p \, d\mu(x). \tag{7.7}$$

3. We show that convergence in the W_p sense implies (7.7). For this we shall use the following elementary inequality: for any $\varepsilon > 0$ there exists a constant $C_\varepsilon > 0$ such that for all nonnegative real numbers a, b,

$$(a + b)^p \leq (1 + \varepsilon) a^p + C_\varepsilon b^p.$$

Combining this inequality with the usual triangle inequality, we see that whenever x_0, x and y are three points in X, one has

$$d(x_0, x)^p \leq (1 + \varepsilon) \, d(x_0, y)^p + C_\varepsilon \, d(x, y)^p.$$

Now let (μ_k) be a sequence of probability measures in $P_p(X)$ such that $W_p(\mu_k, \mu) \longrightarrow 0$, and π_k an optimal transference plan between μ_k and μ. Integrating the last inequality against π_k and using the marginal property, we find that

$$\int d(x_0, x)^p \, d\mu_k(x) \leq (1 + \varepsilon) \int d(x_0, y)^p \, d\mu(y) + C_\varepsilon \int d(x, y)^p \, d\pi_k(x, y).$$

But of course,

$$\int d(x, y)^p \, d\pi_k(x, y) = W_p(\mu_k, \mu)^p \xrightarrow[k\to\infty]{} 0.$$

Therefore,

$$\limsup_{k\to\infty} \int d(x_0, x)^p \, d\mu_k(x) \leq (1 + \varepsilon) \int d(x_0, x)^p \, d\mu(x).$$

Letting $\varepsilon \to 0$, we obtain (7.7).

4. At this stage, to complete the proof we should show that (i) implies the weak convergence of μ_k to μ, and that (iii) implies (i). We claim that

it suffices to prove these facts only under the additional assumption that d is bounded. Indeed, let $\widetilde{d} = \inf(d, 1)$, and let $\widetilde{W}_p$ be the Monge-Kantorovich metric of order p, constructed with $\widetilde{d}$. Of course, $W_p \geq \widetilde{W}_p$, so if we want to prove that convergence in the W_p sense implies weak convergence, we just have to prove that convergence in the $\widetilde{W}_p$ sense implies the weak convergence. Conversely, let us assume that (iii) is satisfied and that μ_k converges in the $\widetilde{W}_p$ sense, and let us check that μ_k really converges in the W_p sense. For this we shall use the elementary inequality (exercise)

$$d(x,y) \leq d(x,y) \wedge R + 2d(x,x_0)1_{d(x,x_0)\geq R/2} + 2d(x_0,y)1_{d(x_0,y)\geq R/2},$$

and its corollary

$$d(x,y)^p \leq C_p\Big([d(x,y)\wedge R]^p + d(x,x_0)^p 1_{d(x,x_0)\geq R/2} + d(x_0,y)^p 1_{d(x_0,y)\geq R/2}\Big)$$

for some constant $C_p > 0$ depending only on p. Let π_k be an optimal transference plan between μ_k and μ for the transportation cost d^p. By the preceding inequality, for $R \geq 1$ we have

$$W_p(\mu_k,\mu)^p = \int d(x,y)^p\, d\pi_k(x,y)$$

$$\leq C_p \int [d(x,y)\wedge R]^p\, d\pi_k(x,y) + C_p \int_{d(x,x_0)\geq R/2} d(x,x_0)^p\, d\pi_k(x,y) + C_p \int_{d(x_0,y)\geq R/2} d(x_0,y)^p\, d\pi_k(x,y)$$

$$\leq R^p\, \widetilde{W}_p^p(\mu_k,\mu) + C_p \int_{d(x_0,x)\geq R/2} d(x_0,x)^p\, d\mu_k(x) + C_p \int_{d(x_0,y)\geq R/2} d(x_0,y)^p\, d\mu(y).$$

Then we conclude that (i) is indeed satisfied, by letting first $k \to \infty$, then $R \to \infty$, and using (iii).

5. From now on we assume that d is bounded, say $d \leq 1$. So all the distances W_p are equivalent, and we just have to prove the theorem in the case $p = 1$. Then we can use the Kantorovich-Rubinstein representation theorem, and (i) reduces to

$$\sup_{\|\varphi\|_{\mathrm{Lip}}\leq 1} \int \varphi\, d(\mu_k - \mu) \xrightarrow[k\to\infty]{} 0. \tag{7.8}$$

Recall that all φ's appearing in (7.8) may be assumed, without loss of generality, to be bounded by 1 in absolute value.

6. Let us assume that $W_1(\mu_k, \mu) \to 0$, and prove that $\mu_k \to \mu$ in the weak sense. What we have to prove is that for all $\varphi \in C_b(X)$,

$$\int \varphi \, d\mu_k \xrightarrow[k\to\infty]{} \int \varphi \, d\mu. \tag{7.9}$$

From (7.8) we know that (7.9) is true if φ is 1-Lipschitz. Replacing φ by $\varphi/\|\varphi\|_{\mathrm{Lip}}$ if $\varphi \neq 0$, we see that it is also true if φ is Lipschitz. We conclude the argument by recalling that in a metric space, any bounded continuous function can be approached from above and below by a sequence of Lipschitz functions. More precisely (recall Statement 4 in Subsection 1.1.7), there exist sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ of Lipschitz functions, uniformly bounded, such that a_n is pointwise increasing in n, b_n is pointwise decreasing in n, and

$$\lim_{n\to\infty} a_n = \varphi = \lim_{n\to\infty} b_n.$$

Then,

$$\limsup_{k\to\infty} \int \varphi \, d\mu_k \leq \liminf_{n\to\infty} \limsup_{k\to\infty} \int b_n \, d\mu_k = \liminf_{n\to\infty} \int b_n \, d\mu = \int \varphi \, d\mu,$$

the last equality being a consequence of Lebesgue's dominated convergence theorem. Similarly, $\liminf_{k\to\infty} \int \varphi \, d\mu_k \geq \int \varphi \, d\mu$, and this concludes the proof of (7.9).

7. Let us assume that $\mu_k \to \mu$ in the weak sense, and prove (7.8). Let x_0 be an arbitrary element of X, and let $\mathrm{Lip}_{1;x_0}(X)$ be the space of all Lipschitz functions φ on X whose Lipschitz constant is at most 1, and such that $\varphi(x_0) = 0$. What we have to prove is

$$\sup_{\varphi\in\mathrm{Lip}_{1;x_0}} \int \varphi \, d(\mu_k - \mu) \xrightarrow[k\to\infty]{} 0.$$

Of course this will imply (7.8) (why?).

From Prokhorov's theorem we know that there exists an increasing sequence of compact sets $(K_n)_{n\geq 1}$ such that for all $n \geq 1$, we have the inequalities $\sup_k \mu_k[K_n^c] \leq 1/n$ and $\mu[K_n^c] \leq 1/n$. Without loss of generality we assume $x_0 \in K_1$. Then, for each $n \geq 1$,

$$\bigl\{\varphi 1_{K_n}; \quad \varphi \in \mathrm{Lip}_{1;x_0}(X)\bigr\}$$

is a subset of $\mathrm{Lip}_{1;x_0}(K_n)$, and by Ascoli's theorem it is a *compact* subset of $C_b(K_n)$ (endowed with the norm of uniform convergence). Hence, for any value of n, from any sequence in $\mathrm{Lip}_{1;x_0}(X)$ we can extract a subsequence which converges uniformly on K_n. By a diagonal argument, we arrive at the following statement: from any sequence $(\varphi_k)_{k\in\mathbb{N}}$ in $\mathrm{Lip}_{1;x_0}(X)$ we can extract a subsequence which converges uniformly on each K_n to some measurable function φ_∞, defined on $S = \bigcup K_n$, which happens to be bounded Lipschitz because the family (φ_k) is uniformly bounded and uniformly Lipschitz.

We apply this statement to a family (φ_k) satisfying

$$\sup_{\varphi\in \mathrm{Lip}_{1;x_0}} \int \varphi\, d(\mu_k - \mu) \leq \int \varphi_k\, d(\mu_k - \mu) + \frac{1}{k}.$$

So there exists a subsequence, still denoted by (φ_k), which converges, uniformly on each K_n, to a 1-Lipschitz function φ_∞ on $S = \bigcup K_n$.

Next, it is a general fact that a 1-Lipschitz function F defined on a subset S of a metric space X can be extended into a 1-Lipschitz function $\widetilde{F}$ on the whole of X. For instance, $\widetilde{F}(x) = \inf_{y\in S}[F(y) + d(x,y)]$ will do. Thus we can extend φ_∞ into an element of $\mathrm{Lip}_{1;x_0}(X)$; in particular φ_∞ is continuous and bounded (because the distance itself is bounded).

To conclude the proof of the theorem it only remains to establish that $\int \varphi_k\, d(\mu_k - \mu)$ goes to 0 as $k \to \infty$. For this we write

$$\int \varphi_k\, d(\mu_k - \mu) \leq \left|\int_{K_n} (\varphi_k - \varphi_\infty)\, d(\mu_k - \mu)\right| + \left|\int_{K_n^c} (\varphi_k - \varphi_\infty)\, d(\mu_k - \mu)\right| + \left|\int_X \varphi_\infty\, d(\mu_k - \mu)\right|.$$

For any given n, the first term in the right-hand side goes to 0 as $k \to \infty$, because φ_k converges uniformly to φ_∞ on K_n as $k \to \infty$. Since all φ_k's and φ_∞ are uniformly (in k) bounded, the second term is bounded by $C(\mu_k[K_n^c] + \mu[K_n^c]) \leq 2C/n$, for some constant $C > 0$; hence it converges to 0 as $n \to \infty$, uniformly in k. Finally, the third term goes to 0 as $k \to \infty$ because μ_k converges weakly to μ. We conclude the proof of Theorem 7.12 by letting first $n \to \infty$, and then $k \to \infty$. □

Exercise 7.14. Write down a simplified version of this proof when X is a compact subset of $\mathbb{R}^n$, endowed with the Euclidean distance.

7.3. The real line

It is interesting to see how Theorem 7.12 can be proven much more directly when X is the real line, in which case everything can be expressed in terms of cumulative distribution functions. Recall the following standard fact, whose proof can be found in almost any introductory textbook in probability theory.

Proposition 7.15 (Weak convergence in terms of cumulative distribution functions). *Let (μ_k) be a sequence of probability measures on $\mathbb{R}$, with respective cumulative distribution functions F_k, and let μ be a probability distribution with cumulative distribution function F. Then, μ_k converges to μ in the weak sense if and only if*

$$F_k(x) \longrightarrow F(x) \quad \text{at each point } x \text{ where } F \text{ is continuous.}$$

Recall also that, by (2.48),

$$W_1(\mu_k, \mu) = \|F_k - F\|_{L^1(\mathbb{R})},$$

if W_1 is computed with the usual distance on the real line. For simplicity let us assume that we deal with probability distributions on $\mathbb{R}$ with common compact support. Taking into account the equivalence of all Monge-Kantorovich distances in this case, Theorem 7.12 translates into the following statement: *let* (F_k) *and* F *be* nondecreasing, *right-continuous functions on an interval* $[a, b] \subset \mathbb{R}$, *with values 0 at* a, *1 at* b. *Then,* $F_k \longrightarrow F$ *in the* L^1 *norm if and only if* $F_k(x) \longrightarrow F(x)$ *at each point* x *where* F *is continuous.*

A drawing is enough to convince one of the truth of this last statement, and it is also easy to understand why nothing can be said at points of discontinuity (exercise).

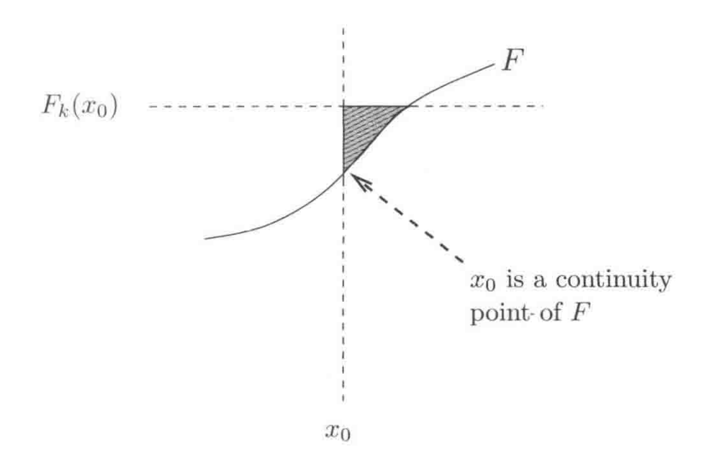


Figure 7.1. $\|F_k - F\|_{L^1}$ is at least equal to the area of the striped region.

Skorohod's representation theorem asserts the following: whenever $(V_k)_{k\in\mathbb{N}}$ is a sequence of real-valued random variables, converging in distribution to a limit random variable V (i.e. the law μ_k of V_k converges weakly to the law μ of V), then there exist random variables $(V'_k)_{k\in\mathbb{N}}$ and V', such that for all k, V'_k has the same law as V_k, and V' has the same law as V, and V'_k converges *almost everywhere* to V'. This theorem can be proven by elementary means: it suffices to set

$$V' = F^{-1}(\mathcal{U}), \qquad V'_k = F_k^{-1}(\mathcal{U}),$$

where $\mathcal{U}$ is uniformly distributed in $[0, 1]$, and to check the criterion given by Proposition 7.15. But with the help of the Wasserstein W_1 distance, it

is possible to give a more quantitative version of Skorohod's theorem, in terms of L^1 norms: with the notation $\mathbf{E}$ standing for expectation, under the additional assumption

$$\lim_{R\to\infty} \limsup_{k\to\infty} \mathbf{E}\, |V_k| 1_{|V_k|\geq R} = 0$$

(which is satisfied for instance if $\sup \mathbf{E}\, |V_k|^p < +\infty$ for some $p > 1$), we know from Theorem 7.12 that

$$W_1(\mu_k, \mu) \longrightarrow 0,$$

and we can write

$$\|V_k' - V'\|_{L^1} = \|F_k^{-1} - F^{-1}\|_{L^1(0,1)} = W_1(\mu_k, \mu).$$

7.4. Behavior under rescaled convolution

We conclude this chapter by discussing some elementary but quite useful "convexity" properties satisfied by the Monge-Kantorovich distances. Of course, it is an immediate consequence of the convexity of the Monge-Kantorovich minimization problem that, for any cost function c, probability measures μ_1, μ_2, ν_1, ν_2 and $\alpha \in [0,1]$,

$$\mathcal{T}_c\big(\alpha\mu_1 + (1-\alpha)\mu_2,\, \alpha\nu_1 + (1-\alpha)\nu_2\big) \leq \alpha\mathcal{T}_c(\mu_1,\nu_1) + (1-\alpha)\mathcal{T}_c(\mu_2,\nu_2).$$

But when X is a *normed space* and the cost function is equal to the power p of the norm, some additional properties hold true. To state them in an appealing way, we shall use the following shorthand: whenever U and V are two random variables with respective law μ, ν, we write

$$\mathcal{T}_p(U,V) = \mathcal{T}_p(\mu,\nu), \qquad W_p(U,V) = W_p(\mu,\nu)$$

for the optimal transportation cost and the Wasserstein distance which are associated with the cost function $\|\cdot\|^p$. By definition, a random variable U is said to lie in L^p if $\mathbf{E}\,\|U\|^p < +\infty$, where $\mathbf{E}$ again stands for expectation.

Proposition 7.16 (Behavior of Wasserstein distances under rescaled convolution). *Let X be a normed space, and $p \geq 1$. Then,*

(i) Whenever the random variables U, V (with values in X) lie in L^p, and $\alpha \in \mathbb{R}$,

$$\mathcal{T}_p(\alpha U, \alpha V) = |\alpha|^p \mathcal{T}_p(U,V), \qquad W_p(\alpha U, \alpha V) = |\alpha| W_p(U,V).$$

(ii) Whenever the random variables U_1, U_2, V_1, V_2 (with values in X) lie in L^p, and $\alpha_1, \alpha_2 \in \mathbb{R}$, if U_1 is independent of U_2 and V_1 is independent of V_2, then

$$\mathcal{T}_p\big(\alpha_1 U_1 + \alpha_2 U_2, \alpha_1 V_1 + \alpha_2 V_2\big) \leq 2^{p-1}\big[|\alpha_1|^p \mathcal{T}_p(U_1,V_1) + |\alpha_2|^p \mathcal{T}_p(U_2,V_2)\big].$$

Proof. We only prove (ii), since the argument for (i) is similar. Here it is convenient to work at the level of the random variables, and write

$$\mathbf{E}\big\|(\alpha_1 U_1 + \alpha_2 U_2) - (\alpha_1 V_1 + \alpha_2 V_2)\big\|^p = \mathbf{E}\big\|\alpha_1(U_1 - V_1) + \alpha_2(U_2 - V_2)\big\|^p$$
$$\leq 2^{p-1}\big(|\alpha_1|^p\,\mathbf{E}\,\|U_1 - V_1\|^p + |\alpha_2|^p\,\mathbf{E}\,\|U_2 - V_2\|^p\big).$$

Here we have used again the elementary inequality $(a+b)^p \leq 2^{p-1}(a^p + b^p)$ (which by the way is a particular case of Hölder's inequality). To conclude the proof we take the infimum on the left-hand side first (with respect to all possible couplings of $\alpha_1 U_1 + \alpha_2 U_2$ and $\alpha_1 V_1 + \alpha_2 V_2$), and then on the right-hand side. □

For $p = 2$, when X is a Hilbert space, the preceding bound can be improved by a factor 2 under a mean value condition, and this is crucial in many applications:

Proposition 7.17 (Subadditivity of $\mathcal{T}_2$ under rescaled convolution). *Let U_1, U_2, V_1, V_2 be L^2 random variables, with values in a Hilbert space X, such that either $\mathbf{E}\,U_1 = \mathbf{E}\,V_1$ or $\mathbf{E}\,U_2 = \mathbf{E}\,V_2$. Let $\alpha_1, \alpha_2 \in \mathbb{R}$, and further assume that U_1 is independent of U_2 and V_1 is independent of V_2. Then,*

$$\mathcal{T}_2\big(\alpha_1 U_1 + \alpha_2 U_2,\, \alpha_1 V_1 + \alpha_2 V_2\big) \leq \alpha_1^2 \mathcal{T}_2(U_1, V_1) + \alpha_2^2 \mathcal{T}_2(U_2, V_2). \tag{7.10}$$

Proof. Without loss of generality, (U_1, V_1) may be chosen independent from (U_2, V_2), since only the laws of $\alpha_1 U_1 + \alpha_2 U_2$ and of $\alpha_1 V_1 + \alpha_2 V_2$ matter in the definition of the left-hand side in (7.10). We expand the square expectation:

$$\mathbf{E}\big\|\alpha_1(U_1 - V_1) + \alpha_2(U_2 - V_2)\big\|^2 = \alpha_1^2\,\mathbf{E}\,\|U_1 - V_1\|^2 + \alpha_2^2\,\mathbf{E}\,\|U_2 - V_2\|^2$$
$$+ 2\alpha_1\alpha_2\,\mathbf{E}\Big\langle U_1 - V_1,\, U_2 - V_2\Big\rangle,$$

and note that the last term vanishes, since by independence it is equal to $2\alpha_1\alpha_2\langle \mathbf{E}\,U_1 - \mathbf{E}\,U_2,\, \mathbf{E}\,V_1 - \mathbf{E}\,V_2\rangle$. Then we pass to the infimum as before. □

Remarks 7.18. (i) If μ_1 is the law of X_1 and μ_2 is the law of X_2, then $\alpha X_1 + (1-\alpha)X_2$ has law

$$(m_\alpha \# \mu_1) * (m_{1-\alpha} \# \mu_2),$$

where $*$ stands for the convolution product and m_λ for the multiplication by λ. This explains our terminology of "rescaled convolution". In terms of densities, if $X = \mathbb{R}^n$, then

$$d\mu(x) = f(x)\,dx \quad \Longrightarrow \quad d[m_\alpha \# \mu](x) = \frac{1}{|\alpha|^n} f\left(\frac{x}{\alpha}\right)\,dx.$$

(ii) The property of subadditivity with respect to rescaled convolution is also enjoyed by several interesting objects appearing in information theory, such as the Fisher information or the Boltzmann-Shannon information, to be

introduced in later chapters. This is also the case for the following functional, studied in [**251**]:

$$d(\mu,\nu) = \sup_{\xi\in\mathbb{R}^n} \frac{|\hat{\mu}(\xi) - \hat{\nu}(\xi)|}{|\xi|^2}, \tag{7.11}$$

where $\hat{\mu}$, $\hat{\nu}$ stand for the respective Fourier transforms of the probability measures μ, ν, namely $\hat{\mu}(\xi) = \int e^{-ix\cdot\xi}\,d\mu(x)$. It is easy to see that d defines a metric on any subspace of $P_2(\mathbb{R}^n)$ made of probability measures with a common mean value.

One among many applications of these convexity properties is the study of some **limit theorems** for sums of rescaled random variables. To explain why the above properties are useful in the context, let us consider a simple situation where $(V_i)_{i\in\mathbb{N}}$ is a sequence of independent, identically distributed (i.e. with the same law) random variables on (say) $\mathbb{R}^n$, with finite variance, and consider

$$S_m = \frac{V_1 + \ldots + V_m}{\sqrt{m}}.$$

Then, S_{2^k} has the same law as

$$\frac{S_{2^{k-1}} + \widetilde{S}_{2^{k-1}}}{\sqrt{2}},$$

where $\widetilde{S}$ stands for an independent copy of S, i.e. an independent variable with the same law as S. Let G be a Gaussian random variable with the same mean and covariance matrix as X_1. It is well known, and easy to check, that G has the same law as

$$\frac{G + \widetilde{G}}{\sqrt{2}},$$

where again $\widetilde{G}$ is an independent copy of G. Applying Proposition 7.17, we deduce that

$$\mathcal{T}_2(S_{2^k}, G) \le \mathcal{T}_2(S_{2^{k-1}}, G).$$

From this decreasing property and a little bit of work, it is not difficult to conclude that

$$W_2(S_m, G) \xrightarrow[m\to\infty]{} 0, \tag{7.12}$$

which is the classical **central limit theorem**.

Remark 7.19. Estimate (7.12) seems to be a slight refinement of the central limit theorem, since the latter only asserts that the law of S_m converges weakly to the law of G. But in fact, since $E|S_m|^2 = E|G|^2$, for all m, weak convergence and convergence in the W_2 sense are equivalent in this context.

This proof of the central limit theorem can be found in Murata and Tanaka [**210**]. It may seem more complicated than the classical proof based on the Fourier transform, but one of its appealing features is precisely that it avoids passing to Fourier space. We refer to [**225**] for many developments, including in particular Berry-Esseen type theorems and "stable limit theorems", which are replacements of the central limit theorem for infinitely divisible laws with infinite variance. Also of great interest are some estimates about the law of large numbers for empirical distributions; see for instance Problem 16 in Chapter 10, and the references mentioned there.

In connection with this, we mention the following problem, which to our knowledge is not solved:

Open Problem 7.20 (Monotonicity along rescaled convolution). *Let $(X_i)_{i\geq 1}$ (resp. X'_i) be a sequence of identically distributed random variables with values in a Hilbert space, lying in L^2, and let $S_m = (X_1+\ldots+X_m)/\sqrt{m}$ (resp. $S'_m = (X'_1+\ldots+X'_m)/\sqrt{m}$). Assume further that $\mathbf{E}\,X_i = \mathbf{E}\,X'_i$. Then the sequence $W_2(S_{2^k}, S'_{2^k})$ is nonincreasing as a function of k; but is the sequence $W_2(S_m, S'_m)$ also nonincreasing as a function of m?*

In the following section, we shall see another application of the particular properties of the W_2 distance in a more unexpected context.

7.5. An application to the Boltzmann equation

In this section, we set forth the work of Tanaka on the problem of trend to equilibrium for the **spatially homogeneous Boltzmann equation with Maxwellian collision kernel**. The crucial properties needed here are the algebraic and topological properties of the quadratic Wasserstein distance. To illustrate this chapter we have chosen the topic of the Boltzmann equation in order to avoid overlapping with the applications studied in [**225**], and also because of the personal interests of the author — who first became acquainted with Wasserstein distance through Tanaka's work. A word of caution is in order: the impact of the Wasserstein distance here, though very interesting, has been somewhat superseded by more recent research. But this is another reason why it seemed a good idea to recall this interesting contribution before it is forgotten.

7.5.1. The Boltzmann equation. The Boltzmann equation is a quite popular model in nonequilibrium statistical mechanics. It belongs to the class of **kinetic models**, in which a gas is studied through the distribution of the velocities of particles. We recall here some basic background about it, and refer to the standard references [**89, 90**] for much more. A very mathematically-oriented introduction can be found in [**262**].

In a typical kinetic description, the unknown is a time-dependent probability distribution $f(t,x,v)$ on the phase space of particles: t stands for time, x for position and v for velocity. Here we shall only consider a very simplified situation in which the dependence of f on x is not taken into account: this is the assumption of **spatial homogeneity**. We shall also assume that the velocity phase is three-dimensional, that there is just one species of particles, and that their mass is normalized to unity. Then the unknown is just a time-dependent probability distribution $f(t,\cdot)$ on $\mathbb{R}^3_v$, which may be thought of as the velocity distribution of a huge amount of particles. Even if we denote it as a probability density, it still makes sense to consider more general probability measures, possibly singular.

Under adequate modelling assumptions (elastic interactions, low density), one can show by half-heuristic, half-rigorous arguments that the probability density should obey the spatially homogeneous Boltzmann equation. The main theoretical assumption behind this is the so-called Boltzmann chaos assumption; roughly speaking, this assumption postulates the noncorrelation of velocities of particles which are just about to collide. The equation is

$$\frac{\partial f}{\partial t} = Q(f,f) = \int_{\mathbb{R}^3} dv_* \int_{S^2} d\sigma B(v-v_*,\sigma)[f'f'_* - ff_*]. \tag{7.13}$$

Here Q is the quadratic **Boltzmann collision operator**, and f f_*, f', f'_* are shorthand for $f(t,v)$, $f(t,v_*)$, $f(t,v'_*)$, $f(t,v'_*)$ respectively. Moreover, v' and v'_* are the velocities of two particles which are just about to collide, in such a way that their velocities after collision will be v and v_*. Before displaying explicit formulas for v', v'_*, let us point out that, due to the Boltzmann chaos assumption, ff_* is proportional to the joint probability of collision events involving a pair of particles with velocities (v,v_*). In the same way, $f'f'_*$ is proportional to the joint probability of collision events involving particles with velocities (v',v'_*).

A "collision" is just the result of a microscopic interaction of two particles which happen to pass by very close to each other. Thus, v', v'_* are obtained as the solution of a classical scattering problem. Since the collisions are elastic, the conservation laws for momentum and kinetic energy in the process of a collision can be written as

$$\begin{cases} v' + v'_* = v + v_*, \\ |v'|^2 + |v'_*|^2 = |v|^2 + |v_*|^2. \end{cases} \tag{7.14}$$

This set of four relations leaves room for two scalar parameters. This is the significance of the parameter $\sigma \in S^2$ in (7.13). Explicitly, one can define

$$\begin{cases} v' = \dfrac{v+v_*}{2} + \dfrac{|v-v_*|}{2}\,\sigma, \\[2ex] v'_* = \dfrac{v+v_*}{2} - \dfrac{|v-v_*|}{2}\,\sigma. \end{cases} \tag{7.15}$$

All these considerations are independent of the precise form of the interaction. But not all values of the parameter σ are equally likely to be "chosen" in a collision: the relative frequency of these values can only be obtained by solving the scattering problem corresponding to our particular interaction. This is the significance of the function B in (7.13), which we will call the **Boltzmann collision kernel**. Up to a multiplicative factor $|v-v_*|$, it coincides with the **cross-section**. For symmetry reasons, it should depend only on the modulus of the relative velocity, $|v-v_*|$, and on the cosine of the deviation angle θ, which is defined by

$$\cos\theta = \left\langle \frac{v-v_*}{|v-v_*|}, \sigma \right\rangle. \tag{7.16}$$

From the mathematical point of view, we already made a tremendous simplification by assuming the gas to be spatially homogeneous; we shall now make a second huge simplification by assuming that the collision kernel depends only on the cosine of the deviation angle. In the terminology of the Boltzmann equation, such a collision kernel is called **Maxwellian**, and one can show that it is natural when particles interact by forces proportional to the reciprocal of the fifth power of the distance (Maxwellian molecules). Under this assumption of a Maxwellian collision kernel, the Boltzmann equation can be rewritten as

$$\frac{\partial f}{\partial t} = Q(f,f) = \int_{\mathbb{R}^3} dv_* \int_{S^2} d\sigma\, b(\cos\theta)\,[f'f'_* - ff_*], \tag{7.17}$$

As explained in [**262**], a necessary and sufficient condition for (7.17) to make sense is

$$\int_{S^2} b(k\cdot\sigma)(1-k\cdot\sigma)\,d\sigma \left(= |S^1| \int_0^{\pi} b(\cos\theta)(1-\cos\theta)\sin\theta\,d\theta \right) < +\infty, \tag{7.18}$$

where k is an arbitrary vector of unit norm. If this condition is satisfied, then one can prove existence and uniqueness of a solution to (7.17), starting from an initial datum f_0 (which can in fact be a probability measure, not necessarily absolutely continuous) with finite variance, or equivalently finite kinetic energy:

$$\int_{\mathbb{R}^3} f_0(v)|v|^2\,dv < +\infty.$$

Moreover, the set of such solutions is stable under weak convergence: if one considers a sequence f_0^k of initial data, converging in the W_2 sense to some f_0, then for any $t \geq 0$ the corresponding family of solutions $f^k(t, \cdot)$ will converge in the W_2 sense to $f(t, \cdot)$, which is the solution of the Boltzmann equation with initial datum f_0.

If (7.18) is replaced by the stronger assumption

$$\int_{S^2} b(k \cdot \sigma)\, d\sigma < +\infty, \tag{7.19}$$

which is known as **Grad's angular cut-off assumption**, then by a change in physical scales it is always possible to reduce to the case when

$$\int_{S^2} b(k \cdot \sigma)\, d\sigma = 1,$$

and equation (7.17) simplifies again to become

$$\frac{\partial f}{\partial t} = Q^+(f, f) - f, \qquad Q^+(f, f) = \int_{\mathbb{R}^3} dv_* \int_{S^2} d\sigma\, b(\cos\theta) f' f'_*. \tag{7.20}$$

Since the Boltzmann equation models elastic collisions, it is natural to expect that the total momentum and kinetic energy, which are respectively

$$\int_{\mathbb{R}^3} f(v) v\, dv, \qquad \int_{\mathbb{R}^3} f(v) \frac{|v|^2}{2}\, dv,$$

are preserved in time by the flow associated with the Boltzmann equation. This is indeed the case. It is easy to check this, at least formally, by using the identities (7.14) and the so-called pre-postcollisional change of variables $(v, v_*, \sigma) \leftrightarrow (v', v'_*, k)$, where $k = (v - v_*)/|v - v_*|$; this change of variable is involutive.

7.5.2. The entropy. One of the main discoveries of Boltzmann was that the H-functional

$$H(f) = \int_{\mathbb{R}^3} f \log f$$

is decreasing in time along solutions of (7.13). This functional measures the quantity of information contained in f, in the sense of the volume of possible **microstates** corresponding to the **macrostate** f. Since we shall encounter entropy several times in the rest of these notes, it is no waste of time to make a short digression in order to give a rough idea of its meaning.

7.5.3. Heuristic introduction of the entropy functional. Let us consider a general phase space for some physical system, and discretize it into a finite number of states, $S_1, \ldots, S_p$. Consider N particles, each of them

being in one of these p states, and to this physical configuration associate the set of frequencies

$$f_i = \frac{N_i}{N}, \qquad i = 1, \dots, p,$$

where N_i is the number of particles in state S_i.

A microscopic description of matter consists in the collection of the states of all particles: we know, say, that particle number k lies in state $S_{i(k)}$, etc. There are of course p^N possible microscopic configurations.

On the other hand, in a macroscopic description of the system, particles can be distinguished only by their state, and such a description is given by the occupation numbers N_i, or equivalently the frequencies f_i. By definition, the **entropy** of a given macroscopic configuration $(f_1, \dots, f_p)$ is the logarithm of the number W of microscopic configurations which are compatible with it. By elementary combinatorics,

$$W = \frac{N!}{N_1! \dots N_p!}.$$

It can be shown that, when all numbers N_i go to infinity while the frequencies f_i converge to some limits (still denoted f_i), then

$$\frac{1}{N} \log W \longrightarrow -\sum_{i=1}^{p} f_i \log f_i.$$

Defining $H(f) = \sum f_i \log f_i$, we see that the quantity $e^{-NH(f)}$ is a quantitative measure of the amount of possible microstates compatible with a given macrostate. Or, in an information-theoretical language, $H(f)$ tells us how much information the knowledge of the macroscopic configuration gives us about the microscopic one.

Clearly, the natural generalization of this object to a continuous space is $H(f) = \int f \log f$. In Boltzmann's theory, this information functional is identified with the negative of the entropy.

7.5.4. ***H* theorem.** After this brief digression, let us present what is known as Boltzmann's H theorem. For simplicity we only consider our restricted framework of spatially homogeneous, Maxwellian molecules, even though the theorem applies in much more generality. Recall that $H(f)$ is well-defined in $\mathbb{R} \cup \{+\infty\}$ as soon as $\int f(v)|v|^2 \, dv < +\infty$.

Proposition 7.21 (Boltzmann's *H* Theorem). *Let $f = f(t, v)$ be a solution of (7.17) with finite kinetic energy, defined on a time-interval $[t_1, t_2]$ $(t_2 > t_1)$. Then, $H(f(t, \cdot))$ is nonincreasing in time. Moreover, it can be constant on $[t_1, t_2]$ only if $f(t, \cdot)$ is a Maxwellian distribution.*

We do not present here the proof of Proposition 7.21, but refer to [**90, 262**] and the references therein for that.

Remarks 7.22. (i) By definition, a Maxwellian distribution, or hydrodynamical state, is a particular Gaussian distribution, of the form

$$M(v) = \rho \frac{e^{-\frac{|v-u|^2}{2T}}}{(2\pi T)^{3/2}}, \qquad \rho, T > 0, \quad u \in \mathbb{R}^3$$

(in dimension n, the $3/2$ exponent should be changed to $n/2$). The parameters ρ, u, T are called respectively the (macroscopic) density, mean velocity and temperature of the Maxwellian. Since we deal with probability densities here, we shall have $\rho = 1$. Moreover, since u and T are uniquely determined by the mean and variance of the probability distribution M, and since these quantities are invariant under the flow associated with the Boltzmann equation, we shall assume without loss of generality that $u = 0$, $T = 1$. Thus the only Maxwellian that we shall be interested in is

$$M(v) = \frac{e^{-\frac{|v|^2}{2}}}{(2\pi)^{3/2}}.$$

(ii) As is natural to expect from the statement of the H theorem, the Maxwellian M is also the probability distribution which maximizes the entropy functional among all probability distributions with given mean and variance.

7.5.5. Trend to equilibrium. From the considerations of the previous subsection, one naturally expects the following to be true: *Let f_0 be an initial probability distribution with zero mean and unit variance. Assume moreover that $H(f_0) < +\infty$. Let $f(t, v)$ the solution of the Boltzmann equation* (7.17), *starting from f_0 at $t = 0$. Then, $f(t, \cdot)$ converges to the Maxwellian M as $t \to \infty$.*

This statement does hold true, and in fact applies to a much larger context; see [**262**] and the references therein for a detailed account of the matter. By a density argument, it is also sometimes possible to prove convergence even when the entropy is infinite at time 0. Our goal in the next subsections is to present another approach, which treats the case of infinite entropy directly, and does not rely on the H theorem, but which on the other hand seems to be limited to the case of of Maxwellian molecules. This method was devised by Tanaka [**210, 249, 250**] in the seventies. One of its advantages is to illustrate the analogy between the trend to equilibrium for the Boltzmann equation with Maxwellian molecules, and the usual central limit theorem. Also it is associated to a quite interesting *stability* theorem for solutions of the Boltzmann equation.

7.5.6. Contraction properties of the Wasserstein distance. The general idea is the following: consider the equilibrium M as a particular (stationary) solution of the Boltzmann equation, and imbed the problem of trend to equilibrium in the more general problem of proving that two arbitrary solutions of the Boltzmann equation become closer and closer as time goes by. The basic result is given by Tanaka's theorem below. As we said above, we consider initial data which are probability measures with zero mean and unit temperature:

$$\int_{\mathbb{R}^3} f_0(v) v \, dv = 0, \qquad \int_{\mathbb{R}^3} f_0(v) \frac{|v|^2}{2} \, dv = \frac{3}{2}.$$

Theorem 7.23 (**Tanaka's theorem**). *Let f_0 and g_0 be two probability measures with zero mean and unit variance, and let $f(t,\cdot)$, $g(t,\cdot)$ be the associated solutions of the Boltzmann equation* (7.17), *with respective initial data f_0, g_0. Then,*

(i) For all $t \geq 0$,

$$W_2(f(t), g(t)) \leq W_2(f_0, g_0).$$

In words, the quadratic Wasserstein distance W_2 is **nonexpansive** *along the Boltzmann flow.*

(ii) $W_2(f(t,\cdot), g(t,\cdot)) \longrightarrow 0$ as $t \to +\infty$.

Proof of Theorem 7.23. The argument below is not given in full detail, but the reader should be able to fill in most of the gaps with a little bit of work.

1. By the uniqueness theorem recalled above, the solution of the Boltzmann equation is stable under perturbations of the collision kernel. Therefore we only need to prove (i) when the collision kernel b satisfies the cut-off assumption (7.19). After this reduction we just have to treat the simpler equation (7.20).

2. Due to the special form of equation (7.20), and the convexity of $W_2(f,g)^2$ as a functional of f and g, it is sufficient (exercise) to establish the functional inequality

$$W_2\big(Q^+(f,f), Q^+(g,g)\big) \leq W_2(f,g). \tag{7.21}$$

We shall show that (7.21) holds true whenever f and g are probability measures such that

$$\int_{\mathbb{R}^3} f(v) v \, dv = \int_{\mathbb{R}^3} g(v) v \, dv.$$

No condition on the variance is needed here; this will come into play only in the proof of part (ii) of the theorem.

3. By convexity again, it is sufficient to prove (7.21) when the collision kernel b is concentrated on a particular angle θ_0, i.e. when $b(\cos\theta)\sin\theta = \delta_{\theta_0}$. Indeed, if we can do this, then we shall recover the general case by writing an arbitrary $b(\cos\theta)\sin\theta$ as an average of Dirac masses, and using Jensen's inequality. So, in the sequel we assume that b is concentrated on the angle θ_0.

4. At this stage, we introduce **Tanaka's representation of the gain term**:

$$Q^+(f,f) = \int_{\mathbb{R}^6} dv\, dv_*\, f f_* \Pi_{v,v_*,\theta} = \mathbf{E}\, \Pi_{V,V_*,\theta_0}, \tag{7.22}$$

where V and V_* are independent random variables of law f, $\mathbf{E}$ stands for expectation, and Π_{v,v_*,θ_0} is the uniform distribution on the circle $C_{c,r,k}$ with center

$$c = \frac{v + v_* + (v - v_*)\cos\theta_0}{2},$$

and radius

$$r = \frac{|v - v_*|}{2}\sin\theta_0,$$

orthogonal to the unit vector

$$k = \frac{v - v_*}{|v - v_*|}.$$

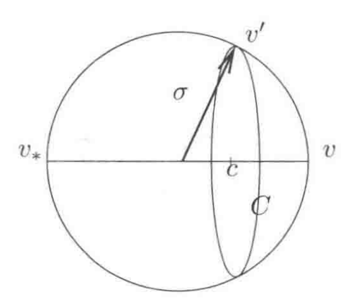


Figure 7.2. $\Pi_{v,v_*,\sigma}$ is the uniform measure on C

The proof of (7.22) is immediate by the use of a weak formulation and again the pre-postcollisional change of variables.

5. We now prove the key estimate.

Lemma 7.24 (Upper bound on circle transport). *Let $U_{c,r,k}$ stand for the uniform probability distribution on the circle $C_{c,r,k}$ with center $c \in \mathbb{R}^3$, radius $r > 0$, orthogonal to $k \in S^2$. Then, one can bound the quadratic*

Wasserstein distance W_2 between two such probability distributions as follows:

$$W\big(U_{c_1,r_1,k_1}, U_{c_2,r_2,k_2}\big) \le |c_1-c_2|^2 + r_1^2 + r_2^2 - r_1 r_2\big(1+|\langle k_1,k_2\rangle|\big). \tag{7.23}$$

Proof. Without loss of generality, assume that $c_1 = 0$, $k_1 = (0,0,1)$ and $k_2 = (0, -\sin\gamma, \cos\gamma)$. On the probability space $\Omega = S^1$, equipped with the uniform probability measure, consider the random variables

$$V_1(\omega) = r_1\,(\cos\omega, \sin\omega, 0),$$

$$V_2(\omega) = c_2 + r_2(\cos\omega, \sin\omega\cos\gamma, \sin\omega\sin\gamma).$$

It is easy to see that V_1 and V_2 have respective laws U_{c_1,r_1,k_1} and U_{c_2,r_2,k_2}. A straightforward computation, and the definition of the W_2 distance, lead to the desired result. □

6. We can now conclude the proof of point (i) in Tanaka's theorem. By Lemma 7.24 and direct computation,

$$W_2\big(\Pi_{v,v_*,\theta}, \Pi_{w,w_*,\theta}\big)^2 \le \left|\frac{1+\cos\theta}{2}(v-w) + \frac{1-\cos\theta}{2}(v_*-w_*)\right|^2$$
$$+\frac{\sin^2\theta}{4}\Big(|v-v_*|^2 + |w-w_*|^2 - |v-v_*||w-w_*| - |\langle v-v_*, w-w_*\rangle|\Big).$$

To plug this into the Tanaka representation of Q^+, we introduce independent random variables V, V_* with law f, and W, W_* with law g (V is independent from V_* and W from W_*, but (V,V_*) is not necessarily independent from (W,W_*)!). By convexity again,

$$W_2\big(Q^+(f,f), Q^+(g,g)\big)^2 = W_2\big(\mathbf{E}\,\Pi_{V,V_*,\theta}, \mathbf{E}\,\Pi_{W,W_*,\theta}\big)^2$$
$$\le \mathbf{E}\,W_2(\Pi_{V,V_*,\theta}, \Pi_{W,W_*,\theta})^2$$

$$\le \mathbf{E}\left|\frac{1+\cos\theta}{2}(V-W) + \frac{1-\cos\theta}{2}(V_*-W_*)\right|^2 \tag{7.24}$$
$$+\frac{\sin^2\theta}{4}\Big(|V-V_*|^2 + |W-W_*|^2 - |V-V_*||W-W_*| - |\langle V-V_*, W-W_*\rangle|\Big).$$

Since

$$\langle V-V_*, W-W_*\rangle \le |\langle V-V_*, W-W_*\rangle| \le |V-V_*||W-W_*|,$$

the second half of (7.24) is bounded by

$$\frac{\sin^2\theta}{4}\Big(|V-V_*|^2 + |W-W_*|^2 - 2\langle V-V_*, W-W_*\rangle\Big)$$
$$= \frac{\sin^2\theta}{4}\,\mathbf{E}\,|(V-W) - (V_*-W_*)|^2.$$

If we expand all the remaining terms in (7.24) and use the independence of the random variables, we find in the end that

$$\mathbf{E}\, W_2(\Pi_{V,V_*,\theta}, \Pi_{V,V_*,\theta})^2 \le \frac{1}{2}(\mathbf{E}\,|V - W|^2 + \mathbf{E}\,|V_* - W_*|^2) = \mathbf{E}\,|V - W|^2.$$

This concludes the proof of (i).

Exercise 7.25. Under the angular cut-off assumption, prove that equality in Tanaka's inequality holds if and only if $f = g$. The following elementary lemma may help: if u is a mapping from $\mathbb{R}^N$ to $\mathbb{R}^N$, $N \ge 2$, such that $u(x) - u(y)$ is always collinear to $x - y$, then u is a multiple of the identity mapping.

7. Now let us consider point (ii) of the theorem. Obviously it is sufficient to prove the special case when $g = M$. Here we shall give the proof only under Grad's cut-off assumption; a simple proof of the general case can be found in Bolley [**51**]. We consider a solution f of the Boltzmann equation, with zero mean and unit variance, and M, the steady Maxwellian solution. By point (i), $W_2(f(t,\cdot), M)$ is nonincreasing, and we have to prove that it actually converges to 0.

In view of part (i) and an easy density argument, it is sufficient to prove the result when $\int f_0(v)|v|^4\, dv < \infty$. Let μ_0 be a weak cluster point of $(f(t,\cdot))_{t\ge 0}$, and consider a subsequence $t_k \to \infty$ such that $f(t_k,\cdot) \longrightarrow \mu_0$ in the weak-$*$ topology of measures.

According to our assumption of bounded fourth moment, it is rather easy to show [**172**] that $\sup_t f(t,v)|v|^4 < \infty$, and in particular

$$\lim_{K\to\infty} \sup_t \int_{|v|\ge K} f(t,v)|v|^2\, dv = 0.$$

So, by Theorem 7.12, $W_2(f(t_k), M) \downarrow W_2(\mu_0, M)$ as $k \to \infty$. But now, we can consider the solution $(\mu(t))_{t\ge 0}$ of the Boltzmann equation with initial value $\mu(0) = \mu_0$. By the weak stability theorem for solutions of the Boltzmann equation, $f(t_k + 1)$ converges weakly to $\mu(1)$, and by the same argument as above, $W_2(f(t_k+1), M) \downarrow W_2(\mu(1), M)$. Since $W_2(f(t), M)$ converges to $W_2(\mu_0, M)$ as $t \to \infty$, it follows that $W_2(\mu(1), M) = W_2(\mu_0, M)$. Thus $W_2(\mu(t), M)$ is stationary for $0 \le t \le 1$. This implies that $\mu_0 = M$, which concludes the proof. □

7.5.7. Bibliographical comments. At the time when it was proven, Tanaka's theorem was new in several respects: it was the first result of weak stability for the Boltzmann equation, and also the first result of trend to equilibrium without the assumption of finite entropy.

Since then, more efficient tools have been devised. In particular, the simpler metric (7.11) was found to be nonexpansive along solutions of the

Boltzmann equation, and was used to derive general and optimal results of uniqueness and stability. As for trend to equilibrium, the advances in entropy estimates have provided specialists with more robust and efficient tools [**252, 253**]. In some refined studies like [**1**], it is shown how to use entropy methods to derive rather precise results, even when the entropy is infinite. In comparison with Tanaka's theorem, entropy methods have the important advantages of covering more general situations (not necessarily Maxwellian) and of leading to explicit estimates.

In spite of this, even from a modern point of view, Tanaka's theorem still has interest; not only for its elegance and simplicity, but also because it reminds us that the increase of entropy is not the only possible explanation for the trend to equilibrium in Boltzmann-like equations.

7.6. Linearization

In this last section, we briefly investigate what becomes of the quadratic Wasserstein distance $W_2(\mu,\nu)$ in the linearized regime when the two probability measures μ and ν are extremely close to each other. Let us start again from Exercise 4.1 about linearization; so we introduce the ansatz

$$\varphi_\varepsilon(x) = \frac{|x|^2}{2} + \varepsilon\psi(x) + O(\varepsilon^2), \qquad \frac{d\nu_\varepsilon}{d\mu} = 1 + \varepsilon h + O(\varepsilon^2),$$

where $\nabla\varphi_\varepsilon\#\mu = \nu_\varepsilon$. This suggests

$$W_2(\mu,\nu_\varepsilon) = \varepsilon\sqrt{\int_{\mathbb{R}^n} |\nabla\psi|^2\,d\mu + O(\varepsilon^2)};$$

moreover, ψ is solution of the Laplace-type equation (4.6), namely $L\psi = h$.

The Laplace-type operator L satisfies the following fundamental integration by parts formula:

$$\int_{\mathbb{R}^n} (Lh_1)h_2\,d\mu = \int_{\mathbb{R}^n} h_1(Lh_2)\,d\mu = \int_{\mathbb{R}^n} \nabla h_1\cdot\nabla h_2\,d\mu.$$

In particular, it is a self-adjoint nonnegative operator. By analogy with the formulas

$$\int_{\mathbb{R}^n} h^2 = \|h\|_{L^2}^2, \qquad \int_{\mathbb{R}^n} |\nabla h|^2 = \|h\|_{\dot{H}^1}^2,$$

we introduce the notation

$$\int_{\mathbb{R}^n} h^2\,d\mu = \|h\|_{L^2(d\mu)}^2, \qquad \int_{\mathbb{R}^n} h(Lh)\,d\mu = \|h\|_{\dot{H}^1(d\mu)}^2. \tag{7.25}$$

These objects define **weighted** square norms, in the sense that they are defined with respect to the reference probability measure $d\mu(x) = f(x)\,dx$

instead of the Lebesgue measure dx. The dot in $\dot{H}^1$ is used to distinguish this space from the usual H^1 space, whose norm is defined by

$$\|h\|_{H^1}^2 = \|h\|_{L^2}^2 + \|h\|_{\dot{H}^1}^2.$$

If the density f of μ is strictly positive everywhere, then the kernel of L is made up of constant functions (why?). Under certain additional regularity conditions which we prefer not to discuss here, one can define the inverse L^{-1} of L on the space of functions h such that $\int h\,d\mu = 0$, and one can establish the formula (which is formally easy to check)

$$\|h\|_{H^{-1}(d\mu)}^2 = \int_{\mathbb{R}^n} h(L^{-1}h)\,d\mu,$$

where by definition

$$\|h\|_{H^{-1}(d\mu)} = \sup\left\{\int_{\mathbb{R}^n} hk\,d\mu; \quad k \in \mathcal{D}(\mathbb{R}^n),\ \|k\|_{\dot{H}^1(d\mu)} = 1\right\}.$$

In words, $H^{-1}(d\mu)$ is the dual space to $\dot{H}^1(d\mu)$.

To sum up, the linearized version of the quadratic Wasserstein distance turns out to be a **weighted H^{-1} norm**. The next theorem will justify part of this statement in a rigorous way.

Theorem 7.26 (From quadratic Wasserstein distance to H^{-1} norm). *Let $\mu \in P_2(\mathbb{R}^n)$ be a probability measure with finite second moment, absolutely continuous with respect to Lebesgue measure; and let $h \in L^\infty(\mathbb{R}^n)$ with $\int h\,d\mu = 0$. Then*

$$\|h\|_{H^{-1}(d\mu)} \le \liminf_{\varepsilon\to 0} \frac{W_2(\mu, (1+\varepsilon h)\mu)}{\varepsilon}. \tag{7.26}$$

Remark 7.27. We shall not here consider the converse of this inequality, which requires more assumptions and more effort.

The first step in the proof of Theorem 7.26 consists in a density argument by which one can reduce to the case when h is smooth and compactly supported. It is the content of the following (somewhat tedious) exercise.

Exercise 7.28. The goal of this exercise is to show that if we can prove Theorem 7.26 under the additional assumption that h is smooth and compactly supported, then we can prove it in the general case.

(i) Let $h \in L^\infty(\mathbb{R}^n)$ satisfy $\int h\,d\mu = 0$. By truncation and regularization by convolution, construct a sequence $(h_j)_{j\in\mathbb{N}}$ of smooth, uniformly bounded, compactly supported functions which converges to h as $j \to \infty$, $d\mu$-almost everywhere [this is the only place where the absolute continuity of μ is useful], in $L^2(d\mu)$ and in $L^1(|x|^2\,d\mu(x))$. Modify the sequence in such a way that it also satisfies the normalization condition $\int h_j\,d\mu = 0$.

(ii) Let $k \in \mathcal{D}(\mathbb{R}^n)$ satisfy $\|k\|_{\dot{H}^1} = 1$. Prove $\int hk\,d\mu = \lim \int h_j k\,d\mu$, and deduce that

$$\|h\|_{H^{-1}(d\mu)} \leq \liminf_{j\to\infty} \|h_j\|_{H^{-1}(d\mu)}.$$

(iii) Show that it is sufficient to prove (7.26) under the additional assumption $h_j \geq -1/2$, $h \geq -1/2$. Under this assumption, show that $(1+h_j)\mu$ converges in the W_2 metric to $(1+h)\mu$. Use this fact to construct a family of very efficient transference plans from $(1+\varepsilon h_j)\mu$ to $(1+\varepsilon h)\mu$, and conclude that

$$\lim_{j\to\infty} \limsup_{\varepsilon\to 0} \frac{W_2\big((1+\varepsilon h_j)\mu, (1+\varepsilon h)\mu\big)}{\varepsilon} \leq \lim_{j\to\infty} W_2\big((1+h_j)\mu, (1+h\mu)\big) = 0.$$

(iv) Using steps (i), (ii) and (iii), show that it is sufficient to prove Theorem 7.26 for smooth, compactly supported h.

Now we just have to prove Theorem 7.26 when h is smooth and compactly supported. The argument which we present below is quite instructive; its core was shown to us by Otto, while Klartag provided some simplification.

Proof of Theorem 7.26 when $h \in \mathcal{D}(\mathbb{R}^n)$. By definition of the norm in H^{-1}, we just have to prove that for any smooth $g \in \mathcal{D}(\mathbb{R}^n)$,

$$\int g\,h\,d\mu \leq \|g\|_{H^1(d\mu)} \liminf_{\varepsilon\to 0} \frac{W_2\big(\mu, (1+\varepsilon h)\mu\big)}{\varepsilon}. \tag{7.27}$$

If $g \in \mathcal{D}(\mathbb{R}^n)$, then there exists a constant C, say $C = 2\|D^2 g\|^2_{L^\infty}$, such that

$$\forall (x,y) \in \mathbb{R}^n \times \mathbb{R}^n \qquad g(x) - g(y) \leq |\nabla g(x)||x-y| + C|x-y|^2. \tag{7.28}$$

Let us use the notation $\mu_\varepsilon = (1+\varepsilon h)\mu$. Then

$$\int_{\mathbb{R}^n} g\,h\,d\mu = \int_{\mathbb{R}^n} g\,d\left(\frac{\mu_\varepsilon - \mu}{\varepsilon}\right).$$

In particular, if π_ε is an optimal transportation plan between μ and μ_ε, then

$$\int_{\mathbb{R}^n} g\,h\,d\mu = \frac{1}{\varepsilon} \int_{\mathbb{R}^n\times\mathbb{R}^n} [g(x) - g(y)]\,d\pi_\varepsilon(x,y).$$

By inequality (7.28), Cauchy-Schwarz and the marginal property of π_ε,

$$\int_{\mathbb{R}^n} h^2\,d\mu \leq \frac{1}{\varepsilon} \int_{\mathbb{R}^{2n}} |\nabla g(x)||x-y|\,d\pi_\varepsilon(x,y) + \frac{C}{\varepsilon} \int_{\mathbb{R}^n\times\mathbb{R}^n} |x-y|^2\,d\pi_\varepsilon(x,y)$$

$$\leq \frac{1}{\varepsilon} \sqrt{\int_{\mathbb{R}^n} |\nabla g(x)|^2\,d\mu(x)} \sqrt{\int_{\mathbb{R}^n\times\mathbb{R}^n} |x-y|^2\,d\pi_\varepsilon(x,y)} + \frac{C}{\varepsilon} \int_{\mathbb{R}^n\times\mathbb{R}^n} |x-y|^2\,d\pi_\varepsilon(x,y)$$

$$= \|g\|_{\dot{H}^1(d\mu)} \left[\frac{W_2(\mu, (1+\varepsilon h)\mu)}{\varepsilon} \right] + C \, \frac{W_2(\mu, (1+\varepsilon h)\mu)^2}{\varepsilon}.$$

Now, if $\limsup W_2(\mu, \mu_\varepsilon)/\varepsilon = +\infty$, then there is nothing to prove; and otherwise the last term of the previous computation converges to 0 as $\varepsilon \to 0$. The desired conclusion follows. □

Chapter 8

A Differential Point of View on Optimal Transportation

In this chapter we shall reformulate the optimal transportation problem in a differential way, inspired by fluid mechanics, which will turn out to be useful for the study of certain dynamical problems.

The following discussion is made only for the quadratic cost $c(x-y) = |x-y|^2$ in $\mathbb{R}^n$. This case is at the same time the most important for applications which we have in mind, and the one with most structure. The scarce existing literature on non-quadratic costs will be reviewed briefly.

Our point of view in this chapter is as follows: most of all, we want to emphasize the formal links between the various objects, but often we will not try to give full justification. One of the best features of these links lies in the insights which they bring on some problems which are apparently not connected with mass transportation. Therefore, even the formal discussion has great interest.

However, as we mention in the sequel, most of the following informal explanations has been put on solid ground in the years following the first edition of this book, at least as far as geometry and gradient flows are concerned (much less so for Hamiltonian flows). Still, let us insist on the heuristic value of these insights, and refrain from going into their tricky rigorous justification. Some of the exercises may give the reader a flavor of the arguments.

8.1. A differential formulation of optimal transportation

The discussion in Chapter 5 about displacement interpolation suggests that there should be a formulation of the optimal transportation problem in terms of fluid mechanics — an "Eulerian" formulation. This is indeed the case, as was pointed out by Benamou and Brenier [**39**].

8.1.1. The Benamou-Brenier reformulation. Let ρ_0 and ρ_1 be two probability densities on $\mathbb{R}^n$, say with compact support; think of them as the density functions of some set of particles at respective times $t = 0$, $t = 1$. Assume that for each time t, there is a velocity field v_t which moves particles around: if $X(t)$ stands for the position of some given particle, then

$$\frac{d}{dt}X(t) = v_t(X(t)). \tag{8.1}$$

If $v = v_t(x)$ is regular enough, for instance uniformly Lipschitz, then by Cauchy-Lipschitz theory it is associated with a well-defined flow on the whole time interval $0 \le t \le 1$: for each x_0 there exists a unique solution $X_{x_0}(t)$ to the equation (8.1), with $X_{x_0}(0) = x_0$; moreover the map $(t, x_0) \longmapsto X_{x_0}(t)$ is globally Lipschitz and one-to-one. Then Theorem 5.34 ensures that the density (ρ_t) of particles at time t evolves as a weak solution of the linear transport equation

$$\frac{\partial \rho_t}{\partial t} + \nabla_x \cdot (\rho_t v_t) = 0. \tag{8.2}$$

At each time t one can define a total **kinetic energy** of particles: up to a factor $1/2$, this is

$$E(t) = \int_{\mathbb{R}^n} \rho_t(x)|v_t(x)|^2\, dx. \tag{8.3}$$

Therefore, to each velocity field is associated an **action**, or time-integral of the energy,

$$A[\rho, v] = \int_0^1 \left(\int_{\mathbb{R}^n} \rho_t(x)|v_t(x)|^2\, dx \right) dt. \tag{8.4}$$

One can think of $A[\rho, v]$ as the total effort which one has to spend in order to move particles around according to the velocity field (v_t).

Now we introduce the

Benamou-Brenier minimization problem:

$$\textit{Minimize} \quad A[\rho, v] \quad \textit{for } (\rho, v) \textit{ in } V(\rho_0, \rho_1), \tag{8.5}$$

where $V(\rho_0, \rho_1)$ *is the set of all* $(\rho, v) = (\rho_t, v_t)_{0 \le t \le 1}$ *such that*

$$(8.6) \quad \begin{cases} \rho \in C\big([0,1];\, w*\!-\!P_{\mathrm{ac}}(\mathbb{R}^n)\big); \\[2ex] v \in L^2(d\rho_t(x)\, dt); \\[2ex] \bigcup_{0 \le t \le 1} \mathrm{Supp}(\rho_t) \quad \textit{is bounded}; \\[2ex] \dfrac{\partial \rho_t}{\partial t} + \nabla \cdot (\rho_t v_t) = 0 \quad \textit{weakly (in the distributional sense)}; \\[2ex] \rho(t=0,\cdot) = \rho_0; \quad \rho(t=1,\cdot) = \rho_1. \end{cases}$$

Here the notation $w*\!-\!P_{\mathrm{ac}}(\mathbb{R}^n)$ stands for the set $P_{\mathrm{ac}}(\mathbb{R}^n)$, endowed with the weak-$*$ topology. Our first main goal here is the following theorem.

Theorem 8.1 (Benamou-Brenier formula). *Let* $\rho_0, \rho_1 \in P_{\mathrm{ac}}(\mathbb{R}^n)$ *be compactly supported. Then, with the notation of* (8.5)–(8.6),

$$\mathcal{T}_2(\rho_0, \rho_1) = \inf\Big\{A[\rho, v];\ (\rho, v) \in V(\rho_0, \rho_1)\Big\}.$$

Remark 8.2. The assumption of compact support which is made in (8.6) and in Theorem 8.1 is too strong, but we will not try to relax it. The main interest of this assumption is to avoid the subtle problems linked to the possibility of characteristic lines going to infinity in finite time, even with a very smooth velocity field. It was checked by Ambrosio, Gigli and Savaré [**13**] that the result still holds true under the sole assumption that ρ_0 and ρ_1 have finite second order moments.

The proof presented here is based on the ideas given in the original paper [**39**], together with a smoothing argument – quite natural for problems of this type. A very slight variant of it can be found in [**13**]. Because the probability measures and velocity fields under consideration are not necessarily smooth, we shall go through some regularization arguments which may obscure the main idea a bit; so the reader is advised to rewrite the core of the proof for himself or herself, leaving apart all the problems of rigorous justification.

Proof of Theorem 8.1. We divide the proof into three steps. Once again we shall abuse notation by identifying measures and densities.

Step 1. Let us prove that

$$\inf\big\{A[\rho, v];\ (\rho, v) \in V_{\mathrm{sm}}(\rho_0, \rho_1)\big\} \ge \mathcal{T}_2(\rho_0, \rho_1),$$

where $V_{\rm sm}$ stands for the set of (ρ, v) in $V(\rho_0, \rho_1)$ such that v is smooth, more precisely bounded and of class C^1. We recall that under our assumptions on ρ_0 and ρ_1,

$$\mathcal{T}_2(\rho_0, \rho_1) = \inf\left\{\int \rho_0(x)|T(x) - x|^2\, dx;\ T\#\rho_0 = \rho_1\right\}. \tag{8.7}$$

Whenever $(\rho, v) \in V_{\rm sm}(\rho_0, \rho_1)$, one can define the associated trajectory flow $T_t(x) = T_t x$ as the solution of $(d/dt)T_t(x) = v_t(T_t(x))$, starting from $T_0 x = x$. From (8.2) we deduce that $\rho_t = T_t\#\rho_0$ (see Theorem 5.34 and the exercises thereafter). In particular,

$$\begin{aligned}\int \rho_t(x)|v_t(x)|^2\, dx &= \int \rho_0(x)|v_t(T_t x)|^2\, dx\\ &= \int \rho_0(x)\left|\frac{d}{dt}T_t x\right|^2\, dx.\end{aligned}$$

If we integrate this in t and use Proposition 5.2, we find that

$$\begin{aligned}A[\rho, v] &= \int \rho_0(x)\left(\int_0^1 \left|\frac{d}{dt}T_t x\right|^2\, dt\right)\, dx\\ &\geq \int \rho_0(x)|T_1 x - T_0 x|^2\, dx\\ &= \int \rho_0(x)|T_1 x - x|^2\, dx.\end{aligned}$$

Since $T_1\#\rho_0 = \rho_1$, this quantity is bounded below by (8.7).

Step 2. This step consists in an approximation argument to reduce the minimization problem to the case of a smooth velocity field, and conclude that

$$\inf\left\{A[\rho, v];\ (\rho, v) \in V(\rho_0, \rho_1)\right\} \geq \mathcal{T}_2(\rho_0, \rho_1). \tag{8.8}$$

First we change variables (ρ, v) for $(\rho, m) = (\rho, \rho v)$. The advantage is that the minimization problem (8.5) is convex with respect to these new variables. Indeed, $\rho|v|^2$ rewrites as $|m|^2/\rho$, and the function $(\rho, m) \longmapsto |m|^2/\rho$ is jointly convex on $\mathbb{R}_+ \times \mathbb{R}^n$; and it is also lower semi-continuous if we use the convention $0/0 = 0$. In the sequel, we shall abuse notation by writing $(\rho, m) \in V(\rho_0, \rho_1)$ instead of $(\rho, v) \in V(\rho_0, \rho_1)$, and similarly $A[\rho, m]$ instead of $A[\rho, v]$.

Let $(\rho, m) \in V(\rho_0, \rho_1)$. By assumption, there exists a ball $B(0, R)$ which contains $\mathrm{Supp}(\rho_t)$ for all t. In particular, m is identically 0 outside of $B(0, R)$. Let $\overline{\rho}$ be a fixed smooth probability distribution, compactly supported in $B(0, R+1)$, bounded from below by a positive number in $B(0, R)$.

The equation $\partial_t \rho + \nabla \cdot m = 0$ implies

$$\frac{\partial \widetilde{\rho}^\delta}{\partial t} + \nabla \cdot \widetilde{m}^\delta = 0,$$

where

$$\widetilde{\rho}^\delta = (1-\delta)\rho + \delta\overline{\rho}, \qquad \widetilde{m}^\delta = (1-\delta)m.$$

In other words, $(\widetilde{\rho}^\delta, \widetilde{m}^\delta) \in V(\widetilde{\rho}_0^\delta, \widetilde{\rho}_1^\delta)$. On the other hand, by convexity,

$$A[\widetilde{\rho}^\delta, \widetilde{m}^\delta] \leq A[\rho, m],$$

Since $\widetilde{\rho}_0^\delta$ converges to ρ_0 as $\delta \to 0$, and similary $\widetilde{\rho}_1^\delta$ converges to ρ_1, it is sufficient to prove that

$$A[\widetilde{\rho}^\delta, \widetilde{m}^\delta] \geq \mathcal{T}_2(\widetilde{\rho}_0^\delta, \widetilde{\rho}_1^\delta). \tag{8.9}$$

Indeed, since the families $\widetilde{\rho}_0^\delta$ and $\widetilde{\rho}_1^\delta$ have their supports uniformly bounded and converge to ρ_0, ρ_1 respectively as $\delta \to 0$, we know that

$$\mathcal{T}_2(\widetilde{\rho}_0^\delta, \widetilde{\rho}_1^\delta) \xrightarrow[\delta \to 0]{} \mathcal{T}_2(\rho_0, \rho_1),$$

and we will conclude that

$$A[\rho, m] \geq \mathcal{T}_2(\rho_0, \rho_1),$$

which is (8.8).

The interest of working only on (8.9) is that all $\widetilde{\rho}_t^\delta$ are uniformly bounded from below on a neighborhood of the support of $\widetilde{m}_t^\delta$. In the sequel, to alleviate notation we shall denote $\widetilde{\rho}^\delta$, $\widetilde{m}^\delta$ by just ρ, m.

Next, we introduce the regularizing kernel

$$r_\varepsilon(t, x) = \frac{1}{\varepsilon^n} r_1\left(\frac{x}{\varepsilon}\right) \frac{1}{\varepsilon} r_2\left(\frac{t}{\varepsilon}\right),$$

where $r_1 \in \mathcal{D}(\mathbb{R}^n)$, $r_2 \in \mathcal{D}(\mathbb{R}_+)$, $r_i \geq 0$, $\int r_i = 1$ $(i = 1, 2)$, $\mathrm{Supp}(r_1) \subset B(0, 1)$, $\mathrm{Supp}(r_2) \subset (0, 1)$. Then we define

$$\rho^\varepsilon = \rho * r_\varepsilon, \qquad m^\varepsilon = m * r_\varepsilon,$$

where the convolution acts on both variables x and t. So,

$$\frac{\partial \rho^\varepsilon}{\partial t} + \nabla \cdot m^\varepsilon = 0 \qquad \text{on } (\varepsilon, 1-\varepsilon) \times \mathbb{R}^n.$$

Moreover, ρ is uniformly bounded from below on a neighborhood of the support of m, and therefore of m^ε, if ε is small enough; it follows that the vector field

$$v_t^\varepsilon = \frac{m_t^\varepsilon}{\rho_t^\varepsilon}$$

is well-defined for all $t \in (\varepsilon, 1-\varepsilon)$ and for all x in $\mathbb{R}^n$ (with the convention $0/0 = 0$ again); and it is a smooth, C^1 function with compact support, in particular uniformly bounded. Therefore, by Step 1, after time-reparametrization,

$$(1-2\varepsilon)\int_{\varepsilon}^{1-\varepsilon}\int \rho_t^{\varepsilon}|v_t^{\varepsilon}|^2\,dx\,dt \geq \mathcal{T}_2(\rho_{\varepsilon}^{\varepsilon}, \rho_{1-\varepsilon}^{\varepsilon}).$$

On the other hand, by convexity of $(\rho, m) \longmapsto |m|^2/\rho$ again,

$$\rho_t^{\varepsilon}|v_t^{\varepsilon}|^2 = \frac{|(\rho_t v_t) * r_{\varepsilon}|^2}{\rho_t * r_{\varepsilon}} \leq \frac{|\rho_t v_t|^2}{\rho_t} * r_{\varepsilon},$$

which implies

$$A[\rho, v] = \int_0^1\int \frac{|\rho_t v_t|^2}{\rho_t}\,dx\,dt \geq \int_{\varepsilon}^{1-\varepsilon}\int \rho_t^{\varepsilon}|v_t^{\varepsilon}|^2\,dx\,dt \geq \mathcal{T}_2(\rho_{\varepsilon}^{\varepsilon}, \rho_{1-\varepsilon}^{\varepsilon}).$$

Letting ε go to 0, and using the continuity of $\mathcal{T}_2$ under weak convergence, one recovers in the end $A[\rho, v] \geq \mathcal{T}_2(\rho_0, \rho_1)$, as desired.

Step 3. We shall show that there exists $(\rho, v) \in V(\rho_0, \rho_1)$ such that $A[\rho, v] = \mathcal{T}_2(\rho_0, \rho_1)$. For this let $T = \nabla\varphi$ be optimal in the Monge-Kantorovich problem (8.7), and set

$$T_t(x) = (1-t)x + tT(x) \equiv \nabla\varphi_t(x),$$

$$\rho_t = T_t\#\rho_0.$$

Recall that, for each $t > 0$, $\nabla\varphi_t^*$ is the inverse of $\nabla\varphi_t$, $d\rho_t$-almost everywhere. This makes it possible to introduce the velocity field (defined $d\rho_t$-almost everywhere)

$$v_t = \left(\frac{d}{dt}T_t\right) \circ T_t^{-1} = (T - \mathrm{Id}\,) \circ T_t^{-1},$$

which is bounded in view of our assumptions on ρ_0 and ρ_1. We define v_t to be 0 whenever $\rho_t = 0$. By mimicking the proof of Theorem 5.37, it is not difficult to check that (ρ_t, v_t) solves (8.2) in the weak sense.

Whenever Φ is a nonnegative measurable function, one has

$$\int \rho_t\,\Phi(v_t)\,dx = \int \rho_0(x)\,\Phi(T(x) - x)\,dx.$$

In particular, choosing $\Phi(v) = |v|^2$, we find that for all t,

$$\int \rho_t(x)|v_t(x)|^2\,dx = \int \rho_0(x)|T(x) - x|^2\,dx = \mathcal{T}_2(\rho_0, \rho_1). \tag{8.10}$$

Thus,

$$A[\rho, v] = \mathcal{T}_2(\rho_0, \rho_1),$$

which was our goal. □

Remarks 8.3. (i) As we already mentioned, the important principle underlying the approximation argument in Step 2 is that the minimization problem (8.5) is convex with respect to the variables ρ and $m = \rho v$ (not with respect to ρ and v!). Also, an alternative strategy to reduce to the case of strictly positive densities is to use mollifiers which are positive everywhere (see [**13**]).

(ii) Such principles of equivalence between the optimal mass transportation problem and a fluid mechanics problem hold in a more general context. In particular, the variant for the cost function $c(x-y) = |x-y|$ is discussed with full precision by Ambrosio [**12**].

(iii) The natural setting for Theorem 8.1 is certainly that of a smooth complete manifold M, instead of just $\mathbb{R}^n$. Steps 1 and 3 of the argument work almost verbatim for this more general situation: the map $T_t(x)$ is replaced by $\exp_x(-t\nabla\psi(x))$, which is a minimizing geodesic (for $d\rho_0$-almost all x) as a function of t; and the corresponding velocity field has constant norm along this geodesic, namely $\|\nabla\psi(x)\| = d(x, \exp_x(-\nabla\psi(x)))$. More delicate, on the other hand, would be the generalization of the smoothing argument in Step 2; although this should certainly be within the range of known regularization techniques for vector fields on manifolds. But, as remarked by Gigli, we do not really need to go through this approximation argument: our manifold M is embedded in $\mathbb{R}^k$ for some k, and then it is possible to consider the transport equation for the density ρ_t on M as a transport equation for the singular measure $\rho_t\delta_M$ in $\mathbb{R}^k$, defined by $\int_{\mathbb{R}^k} \varphi\, d(\rho_t\delta_M) = \int_M \varphi(x)\, d\rho_t(x)$. Similarly, $\rho_t v_t$ can be considered as a singular vector-valued measure. Then Step 2 applies (we did not use absolute continuity there).

One can also devise a proof which does not use the solution of the minimization problem (8.7), but relies on duality and the Hopf-Lax formula. A formal presentation can be found in [**219**, Section 3]; a quite similar argument, also relying on the Hopf-Lax formula, is given in Brenier [**64, 65**]. The duality turns out to be quite powerful here again, although there are a few technical problems. Even if this proof is much less intuitive than the previous one, it is a nice illustration of several of the principles which we have already encountered in these notes, and the underlying principle is useful to know; so we shall sketch the argument briefly. It will be an excellent exercise for the reader to turn this sketch into a fully justified proof. For simplicity, we shall assume in the definition of $V(\rho_0, \rho_1)$ that the densities ρ_t have their support (by assumption) included in a fixed, compact convex set $S \subset \mathbb{R}^n$.

Sketch of proof of Theorem 8.1, again. By the solution of the Monge-Kantorovich problem and Proposition 5.51 we know how to construct an

initial datum u_0 and a solution $(u_t)_{0\le t\le 1}$ of the Hamilton-Jacobi equation $\partial_t u_t + |\nabla u_t|^2/2 = 0$, such that

$$\mathcal{T}_2(\rho_0,\rho_1) = \int u_1\,d\rho_1 - \int u_0\,d\rho_0;$$

and it is easy to show that the infimum of the action functional A is not more than this value (as in Step 3 of the proof above). So all the problem is to show that

$$\inf A[\rho,v] \ge \int u_1\,d\rho_1 - \int u_0\,d\rho_0.$$

We shall show that this holds true for *all* solutions (u_t) of the Hamilton-Jacobi equation. This may look somewhat surprising at first sight, but should be expected in view of Proposition 5.48. To arrive at the result we shall work out a duality argument between $E = C([0,1]\times S)\times C([0,1]\times S)^n$ and $E^* = M([0,1]\times S)\times M([0,1]\times S)^n$.

As we already mentioned, the function $F : (\rho,m) \longmapsto |m|^2/(2\rho)$ is convex and lower semi-continuous on $\mathbb{R}_+\times\mathbb{R}^n$, with the convention $0/0 = 0$. By using Fatou's lemma, one can check that the functional

$$\Theta(\rho,m) = \begin{cases} \displaystyle\int F(\rho,m)\,dt\,dx & \text{if } (\rho,m)\in L^1_+([0,1]\times S)\times L^1([0,1]\times S)^n,\\[2ex] +\infty & \text{else,}\end{cases}$$

is itself convex and lower semi-continuous on E^*. Using the inequality $\alpha\rho + \beta\cdot m \le \rho(\alpha + |\beta|^2/2) + |m|^2/(2\rho)$ (valid for $\rho\in(0,+\infty)$, $\alpha\in\mathbb{R}$, $\beta, m\in\mathbb{R}^n$), one can establish that

$$\Theta^*(\alpha,\beta) = \begin{cases} 0 & \text{if } \alpha + \dfrac{|\beta|^2}{2} \le 0,\\[2ex] +\infty & \text{otherwise.}\end{cases}$$

We now introduce the functional Ξ defined as follows: $\Xi(\rho,m) = 0$ if (ρ,m) is a weak solution of the linear equation $\partial_t\rho + \nabla_x\cdot m = 0$, with boundary conditions ρ_0 and ρ_1 at $t = 0$ and $t = 1$ respectively; and $\Xi(\rho,m) = +\infty$ else. It can be checked that

$$\Xi(\rho,m) = \sup_{\varphi\in\mathcal{D}([0,1]\times S)} \left\{ \int \varphi(1,x)\,d\rho_1(x) - \int \varphi(0,x)\,d\rho_0(x) \right.$$
$$\left. - \int \partial_t\varphi(t,x)\,d\rho(t,x) - \int \nabla\varphi(t,x)\cdot dm(t,x) \right\}.$$

Combining this information with a minimax argument in $E \times E^*$ (using Theorem 1.9, for instance), one deduces that

$$\inf(\Theta + \Xi) = \sup_{\alpha+|\beta|^2/2\leq 0} \inf_{(\rho,m)} \sup_{\varphi} \left\{ \int (\alpha - \partial_t\varphi)\, d\rho + \int (\beta - \nabla_x\varphi)\cdot dm \right.$$
$$\left. + \int \varphi_1\, d\rho_1 - \int \varphi_0\, d\rho_0 \right\}.$$

The formal optimality conditions here are easy to discover: $\alpha = \partial_t\varphi$, $\beta = \nabla_x\varphi$, $\alpha + |\beta|^2/2 = 0$, which suggests using the Hamilton-Jacobi equation. So we would like to use $\varphi(t,x) = u(t,x)$ as a trial function in the above formula, writing

$$\sup_{(\alpha,\beta)} \inf_{(\rho,m)} \sup_{\varphi} (\dots) \quad \geq \quad \int u_1\, d\rho_1 - \int u_0\, d\rho_0,$$

and this would conclude the argument.

However, one has to be careful: it is not a priori allowed to choose α, β and φ as above, since they may not be continuous. What can be done to justify this, is to go through a regularization procedure: if r_ε is a regularizing kernel as in the previous proof of Theorem 8.1, one can check that $u_\varepsilon = u * r_\varepsilon$ satisfies

$$\frac{\partial u_\varepsilon}{\partial t} + \frac{|\nabla u_\varepsilon|^2}{2} \leq 0 \qquad \text{on } (\varepsilon, 1-\varepsilon)\times\mathbb{R}^n,$$

where we have first extended u to a solution of the Hamilton-Jacobi equation on the whole of $[0,1]\times\mathbb{R}^n$. For given ε, we can reparametrize time to take $t = \varepsilon$ and $t = 1-\varepsilon$ as new initial and final times, respectively — just as we did in Step 2 of the proof of Theorem 8.1. Then we can use

$$\alpha = \partial_t u_\varepsilon, \quad \beta = \nabla_x u_\varepsilon, \quad \varphi = u_\varepsilon$$

as trial functions for fixed ε, finally let ε go to 0, and recover the desired conclusion. □

8.1.2. Otto's interpretation. From this point on, our discussion becomes formal and follows the point of view of Otto [**218**], which was motivated by topics in partial differential equations, to be described later.

If we recall that $\mathcal{T}_2$ is the square of the quadratic Wasserstein distance W_2, then the Brenier-Benamou formula looks very much like a geodesic formula in Riemannian geometry; let us explore this idea. To develop the analogy, we would like to define a metric structure $\langle\cdot,\cdot\rangle_\rho$ on each tangent space $T_\rho P$ (tangent to $P(\mathbb{R}^n)$ at ρ), depending smoothly on ρ. This metric structure should define a norm $\|\cdot\|_\rho$ on each $T_\rho P$, in such a way that

$$W_2(\rho_0,\rho_1)^2 = \inf\left\{ \int_0^1 \left\| \frac{d\rho}{dt} \right\|^2_{\rho(t)} dt; \quad \rho(0) = \rho_0,\ \rho(1) = \rho_1 \right\},$$

the infimum being taken over all paths connecting ρ_0 to ρ_1.

How can we think of a tangent vector in $T_\rho P$? This is the time-derivative at time 0 of some trajectory $t \longmapsto \rho(t)$ with $\rho(0) = \rho$. Since $P(\mathbb{R}^n)$ is convex and generates the vector space $M(\mathbb{R}^n)$, we would expect tangent vectors to be all Radon measures with zero integral, "nonnegative at each place where ρ vanishes".

However, in a fluid-mechanics perspective, this is not the right point of view: we would like to see the path $(\rho(t))$ as the time-evolving density of a set of particles moving continuously with velocities v_t. Assuming for instance that the velocity of particles is uniquely determined by their position, it would follow that ρ solves the transport equation $\partial\rho/\partial t + \nabla \cdot (\rho v) = 0$ (Theorem 5.37). So what we would like to see as a tangent space is the space of all probability densities of the form $-\nabla \cdot (\rho v)$. It is rather natural to restrict to those velocity fields for which the kinetic energy is finite, i.e. $\int |v|^2 \, d\rho < +\infty$.

Still formally, it is then natural to define

$$\left\| \frac{\partial\rho}{\partial t} \right\|_\rho^2 = \inf_{v \in L^2(d\rho;\mathbb{R}^n)} \left\{ \int \rho|v|^2; \quad \frac{\partial\rho}{\partial t} + \nabla \cdot (\rho v) = 0 \right\}. \tag{8.11}$$

Here is the intuition behind this formula: let a density of particles ρ be given, and let $\partial\rho/\partial t$ be an "infinitesimal variation" of this probability density, i.e. an element of $T_\rho P$. We can assume that this infinitesimal variation corresponds to particles moving around, but we have no idea of the way these particles move; all we see is the effect on the probability density (going up or down, depending on the places). Let us try to guess the velocity field of particles: there are many possibilities, in fact all vector fields $v : \mathbb{R}^n \to \mathbb{R}^n$ solving the "continuity equation"

$$-\nabla \cdot (\rho v) = \frac{\partial\rho}{\partial t}$$

are compatible with the observed variation of ρ. Among all of these possible vector fields, we try to select the one, if possible, *whose kinetic energy is lowest.*

At least when ρ is smooth and positive, it is easy to characterize a minimizing vector field v_0. Indeed, let v_0 be a minimizer, and let w be a vector field with zero divergence; then, for any $\varepsilon \neq 0$, $v_0 + \varepsilon w/\rho$ is also admissible in the sense that

$$-\nabla \cdot \left[\rho \left(v_0 + \varepsilon \frac{w}{\rho} \right) \right] = \frac{\partial\rho}{\partial t}.$$

Since v_0 is minimizing, we should have

$$\int \rho |v_0|^2 \le \int \rho \left| v_0 + \varepsilon \frac{w}{\rho} \right|^2.$$

Expanding the square, simplifying out terms of order 0 in ε, dividing by ε and letting ε go to 0, one finds that

$$\int v_0 \cdot w = 0.$$

In other words, v_0 should be orthogonal (in the L^2 sense) to the set of divergence-free vector fields; this means that v_0 should be *a gradient*: $v_0 = \nabla u_0$.

Conversely, if all problems of smoothness are washed out, it is reasonable to expect that the elliptic equation

$$-\nabla \cdot (\rho \nabla u_0) = \frac{\partial \rho}{\partial t} \quad \text{(given)} \tag{8.12}$$

admits a unique solution u_0 in a suitable functional space.

For instance, if $\partial\rho/\partial t \in L^2(d\rho)$ and the measure ρ satisfies a **Poincaré inequality**, in the sense that there exists $\lambda > 0$ such that for all $\varphi \in \mathcal{D}(\mathbb{R}^n)$,

$$\int \rho |\nabla \varphi|^2 \ge \lambda \int \rho \left(\varphi - \int \rho \varphi \right)^2, \tag{8.13}$$

then standard theorems on elliptic partial differential equations assert the existence and uniqueness, up to an additive constant, of a solution in the space of u_0's satisfying $\int |\nabla u_0(x)|^2 \, d\rho(x) < +\infty$. We shall come back in more detail to Poincaré inequalities later on, in Chapter 9; here this was just to give a simple example of a sufficient condition for resolution.

Remark 8.4. If we would like to consider problems with a boundary, say when ρ has to live on a bounded open set on Ω, then the natural boundary conditions for (8.12) would be the Neumann boundary conditions $\nabla u_0 \cdot n = 0$ on $\partial\Omega$, with n standing for the unit normal on $\partial\Omega$. This corresponds to a vector field $v_0 = \nabla u_0$ which is tangent to the boundary.

To sum up: at least formally, one can write

$$\begin{cases} W_2(\rho_0, \rho_1)^2 = \inf \left\{ \displaystyle\int_0^1 \left\| \frac{\partial \rho}{\partial t} \right\|^2_{\rho(t)} dt; \quad \rho(0) = \rho_0,\ \rho(1) = \rho_1 \right\}, \\ \left\| \dfrac{\partial \rho}{\partial t} \right\|^2_{\rho} = \displaystyle\int \rho |\nabla u|^2, \qquad -\nabla \cdot (\rho \nabla u) = \dfrac{\partial \rho}{\partial t}. \end{cases} \tag{8.14}$$

8.1.3. Rigorous justification. Here we shall not try to go beyond the formal level outlined in the previous subsection. However, it is possible to develop a formalism in which $P_{\mathrm{ac},2}(\mathbb{R}^n) = P_2 \cap P_{\mathrm{ac}}(\mathbb{R}^n)$ can be equipped with a "differential" structure, in such a way that most of the formal considerations explained here can be justified. This construction was done independently in [**86**] and in [**13**]. In both works it is shown that $P_{\mathrm{ac},2}(\mathbb{R}^n)$, when endowed with the W_2 metric, is a *length space*: one can define the length of a path $(\rho_t)_{0\leq t\leq 1}$ as

$$L[(\rho_t)_{0\leq t\leq 1}] = \sup_{K\geq 1} \sup_{t_1,\dots,t_K} \sum_{i=1}^{K} W_2(\rho_{t_i}, \rho_{t_{i+1}}),$$

with the conventions $t_0 = 0$, $t_{K+1} = 1$; by definition geodesics are those paths for which the length coincides with $W_2(\rho_0, \rho_1)$, and they are said to have constant speed if $W_2(\rho_s, \rho_t) = (t-s)W_2(\rho_0, \rho_1)$ for all $s < t$. Then for any ρ_0, ρ_1 in $P_{\mathrm{ac},2}(\mathbb{R}^n)$ there exists a unique geodesic with constant speed joining ρ_0 and ρ_1, and this geodesic is obtained by displacement interpolation. (So the construction of displacement interpolation has an intrinsic meaning depending on just the choice of distance W_2.) One can also define the tangent space $T_\rho P_{\mathrm{ac},2}(\mathbb{R}^n)$ as the space of all "optimal velocity fields" in $L^2(d\rho)^n$, i.e. the space of all gradients lying in $L^2(d\rho)^n$. It is then possible to give relevant definitions of differentiability, subdifferentiability, etc. Most of the ideas explained heuristically in the sequel, like the concept of gradient flow with respect to the optimal transportation structure, can be defined by means of this construction. To learn more about this, the reader may consult the reference book [**13**], as well as [**267**, Chapter 7].

Exercise 8.5 (Eulerian representation in a nonsmooth setting). The following rather tricky exercise is extracted from [**13**]; it shows that it is almost always possible to define a velocity vector field, for time-dependent probability measures satisfying some regularity assumptions with respect to the time variable. Surprisingly, the absolute continuity of the probability measures will not be necessary, which leads to the possibility of identifying an infinitesimal path starting at ρ_0 with a vector field v, even if ρ_0 is singular.

(i) Let $(\rho_t)_{0\leq t\leq 1}$ be a path in $P_2(\mathbb{R}^n)$ such that $t \longmapsto \rho_t$ is Lipschitz in the W_2 sense, with constant L. Show that for all $\varphi \in \mathrm{Lip}(\mathbb{R}^n)$ with $\|\varphi\|_{\mathrm{Lip}} \leq 1$, the function $t \longmapsto \int \varphi\, d\rho_t$ is itself Lipschitz with constant L. In particular, $(d/dt)\int \varphi\, d\rho_t$ exists for almost all $t \in (0,1)$, and is bounded by L. Let $(\varphi_k)_{k\in\mathbb{N}}$ be a dense sequence in $\mathrm{Lip}_1(B(0,R))$. Show that for almost all $t \in (0,1)$, $(d/dt)\int \varphi_k\, d\rho_t$ exists for all $k \in \mathbb{N}$. Letting $R \to \infty$ and using a diagonal argument, find a dense sequence in $\mathrm{Lip}_1(\mathbb{R}^n) \cap C_0(\mathbb{R}^n)$, still denoted $(\varphi_k)_{k\in\mathbb{N}}$, for which the same differentiability property holds true.

(ii) Let $\varphi \in C^1(\mathbb{R}^n)$ with $\|\varphi\|_{\text{Lip}} \leq 1$, and let $\pi_{s,t}$ denote an optimal transference plan between ρ_s and ρ_t. Show that, for each time t at which $\int \varphi \, d\rho_t$ is differentiable, one has the estimate

$$\begin{aligned} \frac{d}{dt} \int \varphi \, d\rho_t &= \lim_{h \downarrow 0} \frac{1}{h} \int [\varphi(y) - \varphi(x)] \, d\pi_{t,t+h}(x,y) \\ &\leq \lim_{h \downarrow 0} \frac{W_2(\rho_t, \rho_{t+h})}{h} \sqrt{\int |\nabla \varphi(x)|^2 \, d\rho_t(x)} \\ &\leq L \|\nabla \varphi\|_{L^2(d\rho_t)}. \end{aligned}$$

Hint: Define

$$\psi(x,y) = \begin{cases} \dfrac{|\varphi(x) - \varphi(y)|}{|x-y|} & \text{if } x \neq y, \\ \\ |\nabla \varphi(x)| \quad \text{if } y = x. \end{cases}$$

and note that ψ is bounded and upper semi-continuous; in particular, ψ can be approximated from above by a family of bounded continuous functions on $\mathbb{R}^n \times \mathbb{R}^n$. Use this to show that

$$\liminf_{h \downarrow 0} \int \psi(x,y)^2 \, d\pi_{t,t+h}(x,y) \leq \int |\nabla \varphi|^2 \, d\rho_t(x).$$

(iii) Show that $(d/dt) \int \varphi \, d\rho_t$ is really a functional of $\nabla \varphi$, and deduce that this functional is continuous on $L^2(d\rho_t)$, for almost all t, with continuity constant at most L. Deduce that there exists a vector field v_t such that $\|v_t\|_{L^2(\mathbb{R}^n)} \leq L$, and for almost all t, and all k,

$$\frac{d}{dt} \int \varphi_k \, d\rho_t = \int \nabla \varphi_k \cdot v_t \, d\rho_t.$$

Conclude that (ρ_t) is a weak solution of

$$\frac{\partial \rho_t}{\partial t} + \nabla \cdot (\rho_t v_t) = 0.$$

(iv) For each ρ_t, denote by H_t the Hilbert space which is obtained by completing the space of gradients of smooth functions with compact support, with respect to the norm $L^2(d\rho_t; \mathbb{R}^n)$. Show that one can in fact select v_t in H_t.

Remark 8.6. In Exercise 8.5 we assumed $(\rho_t)_{0 \leq t \leq 1}$ to be Lipschitz continuous, but the natural condition would rather be that (ρ_t) be absolutely continuous as a function of t, i.e. (in the present context): for all $\varepsilon > 0$ there exists $\delta > 0$ such that for all $K \in \mathbb{N}$ and $(s_1, \ldots, s_K, t_1, \ldots, t_K)$ in $[0,1]$,

$$\sum_{1 \leq i \leq K} |t_i - s_i| < \delta \implies \sum_{1 \leq i \leq K} W_2(\rho_{t_i}, \rho_{s_i}) \leq \varepsilon.$$

8.2. Differential calculus in $(P(\mathbb{R}^n), W_2)$

The definition in (8.14) formally endows $P(\mathbb{R}^n)$ with a Riemannian metric structure. Indeed, by polarization, one can define the scalar product of two "tangent vectors" $\partial\rho/\partial t_1$ and $\partial\rho/\partial t_2$:

$$\left\langle \frac{\partial \rho}{\partial t_1}, \frac{\partial \rho}{\partial t_2} \right\rangle_\rho = \int \rho \, \langle \nabla u_1, \nabla u_2 \rangle,$$

where u_1 and u_2 solve

$$-\nabla \cdot (\rho \nabla u_1) = \frac{\partial \rho}{\partial t_1}, \qquad -\nabla \cdot (\rho \nabla u_2) = \frac{\partial \rho}{\partial t_2}.$$

In the sequel we shall omit the subscript ρ for the scalar product, but one should keep in mind that the definition of $\langle \cdot, \cdot \rangle$ really depends on the probability ρ. To recall that the scalar product is associated with the quadratic Wasserstein distance, we shall denote it by $\langle \cdot, \cdot \rangle_W$.

From our construction, the quadratic Wasserstein distance W_2 is the geodesic length associated to this Riemannian structure. Moreover, going through the proof of the Brenier-Benamou representation formula, we can guess that geodesics for this structure are nothing but McCann's *displacement interpolation*: whenever ρ_0 and ρ_1 are given, the geodesic path joining ρ_0 and ρ_1 is

$$\rho_t = \big[(1-t)\mathrm{Id} + t\nabla\varphi\big]\#\rho_0,$$

where $\nabla\varphi$ is the optimal map in the Monge-Kantorovich transportation problem from ρ_0 to ρ_1 with quadratic path. Here, as in Chapter 2, we require geodesic paths to have arc length parametrization (or constant speed), which means $W_2(\rho_0, \rho_t) = tW_2(\rho_0, \rho_1)$. In Eulerian formulation, as discussed in Chapter 5, the equation of such a geodesic path $\rho(t,x) = \rho_t(x)$ is

$$(8.15) \qquad \begin{cases} \dfrac{\partial \rho}{\partial t} + \nabla \cdot (\rho v) = 0, \\[2ex] \dfrac{\partial (\rho v)}{\partial t} + \nabla \cdot (\rho v \otimes v) = 0, \end{cases} \qquad v(t=0,\ x) = \nabla\varphi(x) - x.$$

Also, convexity in the sense of this Riemannian structure (which is, by definition, convexity along geodesics) coincides with McCann's displacement convexity.

With the Riemannian structure come basic **calculus rules** for functions defined on $P(\mathbb{R}^n)$; in particular one can formally define a **gradient** and a **Hessian** operator, respectively denoted by grad_W and Hess_W. Here are the formulas which define them: whenever F is a function(al) on $P(\mathbb{R}^n)$, and

whenever $\partial\rho/\partial t$ is a "tangent vector at ρ_0", then

$$\left\langle \operatorname{grad}_W F(\rho_0), \frac{\partial \rho}{\partial t} \right\rangle_W = DF(\rho_0) \cdot \frac{\partial \rho}{\partial t},$$

$$\left\langle \operatorname{Hess}_W F(\rho_0) \cdot \frac{\partial \rho_0}{\partial t}, \frac{\partial \rho_0}{\partial t} \right\rangle_W = \left.\frac{d^2}{dt^2}\right|_{\text{geod}} F(\rho_t),$$

with $d^2/dt^2|_{\text{geod}}$ standing for the second derivative at $t = 0$ of $t \longmapsto F(\rho_t)$ along the geodesic flow starting at ρ_0 with initial derivative $\partial\rho/\partial t$. This point of view will turn out to be a powerful way of computing.

Remark 8.7. Our notation $\langle \cdot, \cdot \rangle_W$, grad_W and Hess_W is justified by the fact that the whole differential structure can be recovered from just the distance W_2, as we explained in subsection 8.1.3.

Important Exercise 8.8 (Explicit expression of the Monge-Kantorovich gradient). Play with the definitions to check that, if all problems of smoothness are left aside, then

$$\operatorname{grad}_W F(\rho) = -\nabla \cdot \left(\rho \nabla \frac{\delta F}{\delta \rho} \right), \tag{8.16}$$

where $\delta F/\delta\rho$ stands for the gradient of the functional $F(\rho)$ with respect to the standard L^2 Euclidean structure. For instance, if $F(\rho) = \int U(\rho)\, dx$, then $\delta F/\delta\rho = U'(\rho)$.

Hint: Write down the definitions of grad_W and $\operatorname{grad}_{L^2}$.

8.3. Monge-Kantorovich induced dynamics

Many differential systems arising in physics can be described by the conjunction of an energy functional U and a linear operation relating the time-derivative of the system with the gradient of the energy. Two well-known classes are

- gradient flow systems: $\dfrac{dX}{dt} = -\operatorname{grad} E(X)$,
- Hamiltonian systems: $\dfrac{dX}{dt} = -J \operatorname{grad} E(X)$, where J is a "Hamiltonian operator", antisymmetric and satisfying certain particular properties (in the language of differential geometry, J should define a *closed* two-form).

The importance of these two classes of systems cannot be overestimated! They arise naturally in all branches of physics, numerical analysis, engineering, etc. An important remark is that sometimes, *a single system may be*

seen as a gradient (resp. Hamiltonian) flow, from several different points of view. For instance, it is well-known that the heat equation,

$$\frac{\partial u}{\partial t} = \Delta u,$$

can be seen as a gradient flow for the energy $E(u) = \|\nabla u\|_{L^2}^2$, if the metric tensor is defined by the L^2 scalar product; or as a gradient flow for the energy $E(u) = \|u\|_{L^2}^2$, if the metric tensor is defined by the H^{-1} scalar product.

8.3.1. Examples of gradient flows. Many well-known equations for probability densities can be recast in the formalism of gradient flows with respect to the optimal transportation differential structure. As one can check from (8.16), one has the following correspondence between *energy functionals* on the one hand, and *gradient flows* with respect to the differential structure induced by optimal transportations on the other hand:

$$\begin{aligned}
E(\rho) &= \int \rho \log \rho, & \frac{\partial \rho}{\partial t} &= \Delta \rho; \\
E(\rho) &= \int \rho \log \rho + \int \rho V, & \frac{\partial \rho}{\partial t} &= \Delta \rho + \nabla \cdot (\rho \nabla V); \\
E(\rho) &= \frac{1}{m-1} \int \rho^m, & \frac{\partial \rho}{\partial t} &= \Delta \rho^m; \\
E(\rho) &= \frac{1}{2} \int \rho(x)\rho(y) W(x-y)\, dx\, dy, & \frac{\partial \rho}{\partial t} &= \nabla \cdot (\rho \nabla (\rho * W)).
\end{aligned}$$

The energy $\int \rho \log \rho$ is Boltzmann's famous H functional, which has the physical meaning of the negative of an entropy, and which we encountered in our presentation of the Boltzmann equation in Chapter 7. As for the partial differential equations above, they are known under the respective names of **heat** equation, **linear Fokker-Planck** equation, **porous medium** equation, and a particular case of so-called **McKean-Vlasov** equations. All of them are classical subjects of study in the theory of partial differential equations.

More sophisticated examples of gradient flows have been studied by Otto. For instance, in [**217**], a model is presented in which the unknown is a *couple* of probability measures. Also, in Brenier [**64**] one can find a "relativistic heat equation" arising as a gradient flow for a modification of the optimal transportation structure.

8.3.2. "Hamiltonian" flows. Whenever $A = A(x)$ is a matrix-valued function on $\mathbb{R}^n$, and $\partial\rho/\partial t$ is a tangent vector, written as $-\nabla \cdot (\rho \nabla u)$, one can define

$$\widetilde{A} \cdot \frac{\partial \rho}{\partial t} = -\nabla \cdot (\rho\, A \nabla u).$$

Of course this amounts to applying A to the vector field. As a corollary, we have

$$\widetilde{A} \cdot \mathrm{grad}_W F(\rho) = -\nabla \cdot \left(\rho\, A \nabla \frac{\delta F}{\delta \rho} \right).$$

This procedure was introduced by Gangbo; it should be noted that it does *not* define an action, in the sense that $\widetilde{A} \circ \widetilde{B}$ does not in general coincide with $\widetilde{AB}$.

When the phase space is $\mathbb{R}^n \times \mathbb{R}^n$, it is possible to use this definition with the Hamiltonian matrix

$$J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}.$$

This defines an antisymmetric operator $\widetilde{J}$ acting on tangent vectors. It is an empirical fact that many interesting equations from physics can be rewritten as

$$\frac{\partial \rho}{\partial t} = -\widetilde{J}\, \mathrm{grad}_W E(\rho),$$

for some energy functional E. We shall call them "Hamiltonian" by abuse of language; however, we make it clear that this terminology is very debatable, since $\widetilde{J}$ does not seem to define a closed two-form.

Here is an example: look at the set of probability densities on $\mathbb{R}^n_x \times \mathbb{R}^n_v$ (think of x as a position variable, and of v as a velocity variable). Whenever $f(x,v)$ is a probability density on $\mathbb{R}^n_x \times \mathbb{R}^n_v$, define

$$E(f) = \int f(x,v) \left[\frac{|v|^2}{2} + V(x) \right] dv\, dx.$$

Then the corresponding flow is the **linear Vlasov equation**

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f - \nabla V(x) \cdot \nabla_v f = 0.$$

On the other hand, if

$$E(f) = \int f(x,v) \frac{|v|^2}{2}\, dv + \frac{1}{2} \int f(x,v) f(y,w) \phi(x-y)\, dv\, dw\, dx\, dy,$$

i.e. the sum of a kinetic energy and an interaction energy, then the associated "Hamiltonian" system is the **nonlinear Vlasov equation**,

$$\begin{cases} \dfrac{\partial f}{\partial t} + v \cdot \nabla_x f - \nabla_x \overline{V}(t,x) \cdot \nabla_v f = 0, \\[2ex] \overline{V} = \phi *_x \rho, \qquad \rho(t,x) = \displaystyle\int_{\mathbb{R}^n} f(t,x,v)\, dv. \end{cases}$$

The meaning of the above equation is that particles evolve within a force field deriving from a potential $\overline{V}$, itself created by the contributions of all

particles. For that reason this equation is often called a **mean-field equation**. The function $\rho(t,x)$ is just the density of particles at the point x (independently of their velocity): in other words, ρ is the marginal of f on $\mathbb{R}^n_x$.

Remark 8.9. Let $N \geq 1$ and let $\mathcal{P}_N$ be the set of all probability measures of the form

$$\mu = \frac{1}{N}\sum_{i=1}^{N} \delta_{(x_i,v_i)}.$$

Each of these measures can be identified with a unique cloud of N points in $\mathbb{R}^n \times \mathbb{R}^n$, up to permutation of the points. It turns out that $\mathcal{P}_N$ is invariant under the action of the flow associated with the Vlasov equation, and induces on $\mathcal{P}_N$ the classical equations of Hamiltonian mechanics,

$$\dot{x}_i = v_i, \qquad \dot{v}_i = -\frac{1}{N}\sum_{j=1}^{N} \nabla V(x_i - x_j).$$

For this equation the Hamiltonian is

$$\sum_{i=1}^{N} \frac{|v_i|^2}{2} + \frac{1}{2N}\sum_{ij} V(x_i - x_j).$$

The flow is well-defined up to permutation of the points, since H itself is invariant under permutation.

Exercise 8.10. Show that the nonlinear Vlasov equation is also the Hamiltonian equation associated with the energy functional

$$E(f) = \int f(x,v)\,\log f(x,v)\,dv\,dx + \int f(x,v)\frac{|v|^2}{2}\,dv\,dx + \frac{1}{2}\int f(x,v)f(y,w)\phi(x-y)\,dv\,dw\,dx\,dy. \tag{8.17}$$

Another example is provided by the **two-dimensional incompressible Euler equation** in an open subset Ω of $\mathbb{R}^2$, in **vorticity** formulation. The vorticity is the curl of the velocity field, and in two dimensions can be identified with a scalar quantity, which we will here assume to be nonnegative and denote by ρ. Then the two-dimensional incompressible Euler equation is just

$$\frac{\partial \rho}{\partial t} = \nabla\cdot(\rho\nabla^{\perp}\Delta^{-1}\rho),$$

where $\Delta^{-1}\rho$ is the solution V of the Laplace equation $\Delta V = \rho$, with Dirichlet boundary condition $V = 0$ on the boundary of Ω. Moreover, $\nabla^{\perp} f$ is obtained

from ∇f by rotation through an angle $\pi/2$. If $\Omega = \mathbb{R}^2$, then $\nabla^\perp \Delta^{-1}\rho = \Psi * \rho$, where

$$\Psi(z) = \frac{z^\perp}{2\pi|z|^2}, \qquad z^\perp = (-z_2, z_1);$$

this formula for reconstructing $\nabla^\perp \Delta^{-1}\rho$ in terms of ρ is often called the (two-dimensional) **Biot-Savart** law.

When ρ is a probability density, this equation enters our general framework as the Hamiltonian system associated with the energy functional $E(\rho) = \|\rho\|^2_{H^{-1}(\Omega)} = \|\nabla\Delta^{-1}\rho\|^2_{L^2(\Omega)}$. It is interesting to note that the gradient flow version of this equation,

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot (\rho \nabla \Delta^{-1}\rho),$$

appears in the study of Ginzburg-Landau dynamics (see for instance [**192**] for a presentation of the equation, and [**14, 16**] for explicit studies in the context of optimal transportation); while the "antigradient flow" version (or gradient flow for the opposite of the energy),

$$\frac{\partial \rho}{\partial t} = \nabla \cdot (\rho \nabla \Delta^{-1}\rho),$$

appears in the modelling of dynamics of agglomerating particles in two dimensions [**211**] (with loose links to the one-dimensional sticky particle system).

Here are some further examples. Let Ω be a bounded open subset of $\mathbb{R}^2$, with Lebesgue measure normalized to unity, and let 1 stand for the Lebesgue measure on Ω. As explained in Benamou and Brenier [**38**] (see Problem 9 in Chapter 10), the energy $E(\rho) = W_2(\rho, 1)^2$, defined on $P(\Omega)$, leads to an extremely interesting Hamiltonian system in fluid mechanics: the **semi-geostrophic system**. There are variants like $E(\rho) = W_2(\rho,1)^2 + \int \rho^\gamma(x)\,dx$, where $\gamma > 1$ is a physical constant depending on the thermodynamical properties of the fluid; or $E(\rho) = \mathcal{T}_c(\rho, 1) + \int \rho^\gamma(x)\,dx$; etc. For all this we warmly recommend the nice papers by Cullen and Gangbo [**106**], Cullen and Maroofi [**107**].

Let us finally give an example of a flow which is neither gradient, nor Hamiltonian, but somewhat "in between": choose the matrix

$$A = \begin{pmatrix} 0 & -I_n \\ I_n & I_n \end{pmatrix}$$

in conjunction with the energy E appearing in (8.17). Then, the equation $\partial f/\partial t = -\widetilde{A}\,\mathrm{grad}_W\, E(f)$ turns out to be the nonlinear **Vlasov-Fokker-Planck equation**, well-known in plasma physics:

$$\begin{cases} \dfrac{\partial f}{\partial t} + v\cdot\nabla_x f - \nabla_x \overline{V}(t,x)\cdot\nabla_v f = \Delta_v f + \nabla_v\cdot(fv), \\ \\ \overline{V} = V *_x \rho, \qquad \rho(t,x) = \int_{\mathbb{R}^n} f(t,x,v)\,dv. \end{cases}$$

Exercise 8.11 (Two-dimensional Navier-Stokes equation). Find a matrix A and a functional E on $P(\mathbb{R}^2)$ such that $\partial\rho/\partial t = -\widetilde{A}\cdot\mathrm{grad}_W\, E(\rho)$ is the two-dimensional incompressible Navier-Stokes equation in vorticity formulation, i.e. the same as the Euler equation but with a diffusion term $\Delta\rho$ added on the right-hand side.

In the sequel, we shall only focus on gradient flows, which as a general rule are simpler than the other classes of equations described here, and have been studied more systematically in the context of optimal transportation.

8.4. Time-discretization

There are two main ways to make sense of the fact that some partial differential equation is the gradient flow of some energy with respect to the optimal transportation structure. One is to use rigorous notions of subdifferential, and decide that a gradient flow is an equation of the form $d\rho_t/dt \in \partial^- F(\rho_t)$, where ∂^- stands for some appropriate notion of subdifferential. This was considered in [**13**], for instance. The authors showed that the solution to such an equation is unique if F is convex and satisfies some kind of "coercivity" assumption; and they could identify several of the equations mentioned before as gradient flows in this sense.

Another strategy is to go via a time-discretization. This was the approach first used by Jordan, Kinderlehrer and Otto [**173**]. It does not require any study of tangent spaces, subdifferentiability or related concepts, and in fact was developed before the idea that the Wasserstein distance was associated with a Riemannian formalism.

8.4.1. Abstract discretized gradient flows. A classical way to implement gradient flows without making explicit use of the gradient operator, or even without relying on Riemannian structure, consists in introducing a **time-discretization** and passing to the limit as the time step goes to 0.

Here is the general scheme to approximate the gradient flow for some energy functional E in an abstract metric space endowed with a metric, denoted dist. Introduce a time-step $\tau > 0$, and define a sequence $(X_\tau^n)_{n\geq 0}$ inductively as follows: X_τ^0 is the initial datum X^0; given X_τ^n, define X_τ^{n+1}

as the solution (or a solution, if there is no uniqueness) of the minimization problem

$$\min\left[E(X)+\frac{\operatorname{dist}(X_\tau^n,X)^2}{2\tau}\right].$$

Note that this scheme is only expressed in terms of the functional and the distance. In the Euclidean case, the Euler-Lagrange equation corresponding to this problem is of course

$$\frac{X_\tau^{n+1}-X_\tau^n}{\tau}=-\operatorname{grad}E(X_\tau^{n+1}), \tag{8.18}$$

which looks exactly like a time-discretized version of the gradient flow. Next, one defines X_τ on $\mathbb{R}_+$ as the piecewise constant function with value X_τ^n on $[n\tau,(n+1)\tau)$, and tries to pass to the limit as $\tau\to 0$. This limit is called a (generalized) gradient flow. Such objects have been studied in an abstract setting by Ambrosio [**11**].

In order to show that the limit exists as $\tau\to 0$ and satisfies the right equation, one should look for suitable estimates. Let us sketch here the proofs of the *three basic estimates for discretized gradient flows.* For simplicity we assume that E is bounded below by an absolute constant.

First, since X_τ^{n+1} is a minimizer of $X\longmapsto E(X)+\operatorname{dist}(X_\tau^n,X)^2/2\tau$, it does better with respect to this functional than X_τ^n does. In other words,

$$E(X_\tau^{n+1})+\frac{\operatorname{dist}(X_\tau^n,X_\tau^{n+1})^2}{2\tau}\le E(X_\tau^n). \tag{8.19}$$

This has two consequences. The first is the **energy estimate**,

$$\sup_{n\ge 0}E(X_\tau^n)\le E(X^0). \tag{8.20}$$

The second, which is obtained by summing together the inequalities (8.19), is the **total square distance estimate**

$$\sum_{n\ge 0}\operatorname{dist}(X_\tau^n,X_\tau^{n+1})^2\le 2\tau(E(X^0)-\inf E). \tag{8.21}$$

From this last estimate one also deduces an approximate Hölder-1/2 estimate on (X_τ): whenever $s<t$ one has

$$\begin{aligned}\operatorname{dist}(X_\tau(s),X_\tau(t))^2&\le\left[\frac{t-s}{\tau}+1\right]\sum_{\frac{s}{\tau}\le n\le\frac{t}{\tau}}\operatorname{dist}(X_\tau^n,X_\tau^{n+1})^2\\&\le C\big[(t-s)+\tau\big],\end{aligned} \tag{8.22}$$

where $C=2[E(X^0)-\inf E]$; the inequality in (8.22) is a consequence of the triangle inequality for W_2 and the Cauchy-Schwarz inequality.

The next and final estimate takes better advantage of the minimizing property of X_τ^{n+1}. For this estimate we shall assume that there is a nice

underlying Riemannian structure and that the energy functional is smooth. Let w be an arbitrary tangent vector at X_τ^{n+1}, and let us introduce a path $\widetilde{X}_\varepsilon$, for small ε, in such a way that

$$\widetilde{X}_0 = X_\tau^{n+1}, \qquad \left.\frac{d\widetilde{X}_\varepsilon}{d\varepsilon}\right|_{\varepsilon=0} = w.$$

By the minimizing property of X_τ^{n+1}, one has

$$(8.23)\quad E(X_\tau^{n+1}) + \frac{\operatorname{dist}(X_\tau^n, X_\tau^{n+1})^2}{2\tau} \le E(\widetilde{X}_\varepsilon) + \frac{\operatorname{dist}(X_\tau^n, \widetilde{X}_\varepsilon)^2}{2\tau} \qquad \forall \varepsilon > 0.$$

Now, if E is smooth, then

$$(8.24)\qquad E(\widetilde{X}_\varepsilon) = E(X_\tau^{n+1}) + \varepsilon \left\langle \operatorname{grad} E(X_\tau^{n+1}), w\right\rangle + O(\varepsilon^2).$$

On the other hand,

$$\begin{aligned}
&\operatorname{dist}(X_\tau^n, \widetilde{X}_\varepsilon)^2 - \operatorname{dist}(X_\tau^n, X_\tau^{n+1})^2\\
&= \left[\operatorname{dist}(X_\tau^n, \widetilde{X}_\varepsilon) + \operatorname{dist}(X_\tau^n, X_\tau^{n+1})\right]\left[\operatorname{dist}(X_\tau^n, \widetilde{X}_\varepsilon) - \operatorname{dist}(X_\tau^n, X_\tau^{n+1})\right]\\
&\le \left[\operatorname{dist}(X_\tau^n, \widetilde{X}_\varepsilon) + \operatorname{dist}(X_\tau^n, X_\tau^{n+1})\right]\operatorname{dist}(X_\tau^{n+1}, \widetilde{X}_\varepsilon)\\
&\le [2\operatorname{dist}(X_\tau^n, X_\tau^{n+1}) + \operatorname{dist}(X_\tau^{n+1}, \widetilde{X}_\varepsilon)]\operatorname{dist}(X_\tau^{n+1}, \widetilde{X}_\varepsilon).
\end{aligned}$$

Since the distance between X^{n+1} and $\widetilde{X}_\varepsilon$ is $\varepsilon\|w\| + o(\varepsilon)$, one deduces that

$$(8.25)\quad \frac{\operatorname{dist}(X_\tau^n, \widetilde{X}_\varepsilon)^2}{2\tau} \le \frac{\operatorname{dist}(X_\tau^n, X_\tau^{n+1})^2}{2\tau} + \varepsilon \operatorname{dist}(X_\tau^n, X_\tau^{n+1})\frac{\|w\|}{\tau} + o(\varepsilon).$$

Combining (8.24) and (8.25) with (8.23), in the limit $\varepsilon \to 0$, one finds that

$$\langle \operatorname{grad} E(X_\tau^{n+1}), w\rangle + \frac{\operatorname{dist}(X_\tau^n, X_\tau^{n+1})\|w\|}{\tau} \ge 0.$$

The choice $w = -\operatorname{grad} E(X_\tau^{n+1})$ yields, after simplification, the estimate

$$\|\operatorname{grad} E(X_\tau^{n+1})\|^2 \le \frac{\operatorname{dist}(X_\tau^n, X_\tau^{n+1})^2}{\tau^2}.$$

By combining this with the square distance estimate, one finally arrives at the **energy gradient estimate**

$$\tau \sum_{n\ge 0} \|\operatorname{grad} E(X_\tau^n)\|^2 \le 2[E(X^0) - \inf E].$$

In continuous version, this can be rewritten as

$$(8.26)\qquad \int_0^{+\infty} \|\operatorname{grad} E(X_\tau(t))\|^2\, dt \le 2[E(X^0) - \inf E].$$

This estimate actually holds even in a non-Riemannian setting, with a suitable definition of the "norm of the gradient".

The energy estimate, the total square distance estimate and the energy gradient estimate are usually enough to ensure the relative compactness of

the family (X_τ) as $\tau \to 0$. On the other hand, more work is required to pass to the limit as $\tau \to 0$. Usually the main idea is to apply the same procedure as in the energy gradient estimate (by letting X_ε^{n+1} undergo a small perturbation), but to compute a more precise expansion of the variation in dist $(X_\tau^n, \widetilde{X}_\varepsilon)$, and to keep w arbitrary. In this way one obtains an approximate Euler-Lagrange equation, in which one may hope to pass to the limit. We exemplify this in the next subsection.

8.4.2. Application to the setting of optimal transportation. In the context of Monge-Kantorovich induced dynamics, this strategy was first implemented in a very inspiring paper by Jordan, Kinderlehrer and Otto [**173**] for the linear Fokker-Planck equation. Of course, it is much simpler to construct solutions of the Fokker-Planck equation by other means, but this way of proceeding gives interesting insights. Kinderlehrer and Walkington [**177**] showed on some examples that this scheme can be implemented numerically. Moreover, in certain more complicated situations, nontrivial existence results for gradient flows can be obtained in this manner, see for instance Otto [**216**].

Let us explain the construction of [**173**], which follows the general abstract strategy of subsection 8.4.1. In the sequel we do not give full justification, but try to point out the important steps of the argument. To follow the discussion, the reader may need a bit of familiarity with function spaces.

Consider the energy functional

$$E(\rho) = \int \rho \log \rho + \int \rho V$$

on $P(\mathbb{R}^k)$. Here $V(x)$ is a nice smooth potential, satisfying a growth condition, say $V(x) = O(|x|^2)$ at infinity. Let ρ^0 be a probability density taken as initial datum, with $E(\rho^0) < +\infty$. Let $\tau > 0$ be a time step, and introduce the sequence (ρ_τ^n) defined as follows. First, $\rho_\tau^0 = \rho^0$; next, given ρ_τ^n, one constructs ρ_τ^{n+1} as the unique minimizer of

$$\rho \longmapsto E(\rho) + \frac{W_2(\rho_\tau^n, \rho)^2}{2\tau}.$$

Note that the functional E is strictly convex in the usual sense (while displacement convexity a priori holds only if V is convex). This, together with the convexity of the square Wasserstein distance, ensures the uniqueness of the minimizer. Of course,

$$\int \rho_\tau^{n+1} \log \rho_\tau^{n+1} + \int \rho_\tau^{n+1} V + \frac{W_2(\rho_\tau^n, \rho_\tau^{n+1})^2}{2\tau} \leq \int \rho_\tau^n \log \rho_\tau^n + \int \rho_\tau^n V.$$

Reasoning as in the previous section, one obtains a uniform control on $\sup_{t \geq 0} E(\rho_\tau(t))$, which ensures the tightness of the family $(\rho_\tau)_{\tau > 0}$ in the

x variable, as well as the weak-L^1 compactness. Moreover, one gets an approximate Hölder-1/2 continuity estimate in the quadratic Wasserstein distance which ensures the approximate uniform equicontinuity in time of the family $(\rho_\tau)_{\tau>0}$. All this is sufficient to guarantee that, after possible extraction of a subsequence $(\tau_k)_{k\geq 0}$, the family (ρ_τ) converges to some function $\rho : \mathbb{R}_+ \mapsto P_{\mathrm{ac}}(\mathbb{R}^n)$, continuous when $P_{\mathrm{ac}}(\mathbb{R}^n)$ is endowed with the weak L^1 topology. One can also prove a square gradient estimate and obtain a control on

$$\sum_{n\geq 0} \tau \int \rho_\tau^n \Big|\nabla(\log \rho_\tau^n + V)\Big|^2 \, dx = \int_0^{+\infty} \int \rho_\tau(t) \Big|\nabla(\log \rho_\tau(t) + V)\Big|^2 \, dx \, dt,$$

which actually ensures the *strong* L^1 compactness of the family (ρ_τ). But in our context this refinement can be dispended with.

Now, it remains to pass to the limit in the equation. For this we shall introduce a suitable variation for ρ_τ^{n+1}. In a context of mass transportation, it is natural to proceed as follows: let ξ be a smooth vector field with compact support, and let T_ε be the family of trajectories associated with the vector field $\mathrm{Id} + \varepsilon\xi$. Then define

$$\widetilde{\rho}_\varepsilon = T_\varepsilon \# \rho_\tau^{n+1}.$$

For ε small enough, T_ε is a C^1 diffeomorphism, as can be seen as a consequence of the fixed point theorem (exercise), and $\det(\nabla T_\varepsilon) > 0$. Thanks to the definition of $\widetilde{\rho}_\varepsilon$, we have (just as in Section 5.2 of Chapter 5)

$$E(\widetilde{\rho}_\varepsilon) = \int \widetilde{\rho}_\varepsilon \log \widetilde{\rho}_\varepsilon + \int \widetilde{\rho}_\varepsilon V \tag{8.27}$$

$$= \int \rho_\tau^{n+1} \log \frac{\rho_\tau^{n+1}}{\det(I_k + \varepsilon\nabla\xi)} + \int \rho_\tau^{n+1}(x) V(x + \varepsilon\xi(x)) \, dx. \tag{8.28}$$

On the other hand, since ρ_τ^n, ρ_τ^{n+1} are absolutely continuous (thanks to the energy estimate), there exists an optimal map $\nabla\varphi$ such that $\nabla\varphi\#\rho_\tau^n = \rho_\tau^{n+1}$ and

$$W_2(\rho_\tau^n, \rho_\tau^{n+1})^2 = \int \rho_\tau^n(x) |x - \nabla\varphi(x)|^2 \, dx.$$

Then $\widetilde{\rho}_\varepsilon = [(\mathrm{Id} + \varepsilon\xi) \circ \nabla\varphi]\#\rho_\tau^n$; so, by definition of the quadratic Wasserstein distance,

$$W_2(\rho_\tau^n, \widetilde{\rho}_\varepsilon)^2 \leq \int \rho_\tau^n(x) \Big| x - \nabla\varphi(x) - \varepsilon\, \xi \circ \nabla\varphi(x) \Big|^2 \, dx.$$

Putting everything together, we obtain

$$\frac{W_2(\rho_\tau^n, \widetilde{\rho}_\varepsilon)^2}{2\tau} + E(\widetilde{\rho}_\varepsilon) - \frac{W_2(\rho_\tau^n, \rho_\tau^{n+1})^2}{2\tau} - E(\rho_\tau^{n+1})$$

$$\leq \int \rho_\tau^n(x) \left(\frac{|x - \nabla\varphi(x) - \varepsilon\, \xi \circ \nabla\varphi(x)|^2 - |x - \nabla\varphi(x)|^2}{2\tau} \right) dx$$

$$+ \int \rho_\tau^{n+1}(x) \big[V(x + \varepsilon\xi(x)) - V(x) \big]\, dx - \int \rho_\tau^{n+1}(x) \log\det(\mathrm{Id} + \varepsilon\nabla\xi(x))\, dx,$$

and the left-hand side has to be nonnegative because of the minimizing property of ρ_τ^{n+1}. After dividing the whole expression by $\varepsilon > 0$ and letting ε go to 0^+, one finds that

$$0 \leq \frac{1}{\tau} \int \rho_\tau^n(x) \Big\langle \nabla\varphi(x) - x,\, \xi \circ \nabla\varphi(x) \Big\rangle\, dx + \int \rho_\tau^{n+1}(x) \,\langle \nabla V(x), \xi(x) \rangle\, dx$$
$$- \int \rho_\tau^{n+1}(x) (\nabla \cdot \xi)(x)\, dx.$$

Since ξ can be changed into $-\xi$, in fact

$$(8.29)\quad \frac{1}{\tau} \int \rho_\tau^n(x) \Big\langle \nabla\varphi(x) - x,\, \xi \circ \nabla\varphi(x) \Big\rangle\, dx$$
$$= \int \rho_\tau^{n+1}(x) \Big[(\nabla \cdot \xi)(x) - \langle \nabla V(x), \xi(x) \rangle \Big]\, dx.$$

Assume now that $\xi = \nabla\zeta$, for some $\zeta \in \mathcal{D}(\mathbb{R}^n)$. From the expansion

$$\zeta(\nabla\varphi(x)) - \zeta(x) = \Big\langle \nabla\varphi(x) - x,\, \nabla\zeta \circ \nabla\varphi(x) \Big\rangle + O(|x - \nabla\varphi(x)|^2)$$

one deduces that the left-hand side in (8.29) can be recast as

$$\frac{1}{\tau} \left(\int \rho_\tau^n(x)\, \zeta \circ \nabla\varphi(x)\, dx - \int \rho_\tau^n(x) \zeta(x)\, dx \right)$$
$$+ O\left(\frac{1}{\tau} \int \rho_\tau^n(x) |x - \nabla\varphi(x)|^2\, dx \right)$$
$$= \frac{1}{\tau} \left(\int \rho_\tau^{n+1} \zeta - \int \rho_\tau^n \zeta \right) + O\left(\frac{W_2(\rho_\tau^n, \rho_\tau^{n+1})^2}{\tau} \right).$$

If one plugs this into (8.29) and sums from $n_1 = [t_1/\tau]$ to $n_2 = [t_2/\tau] + 1$, where t_1 and t_2 are two arbitrary times, then one finds that

$$\int \rho_\tau(t_2)\zeta - \int \rho_\tau(t_1)\zeta + O\left(\sum_{n=n_1}^{n_2} W_2(\rho_\tau^n, \rho_\tau^{n+1})^2 \right)$$
$$= \int_{t_1}^{t_2} \int \rho_\tau(t) (\Delta\zeta - \nabla V \cdot \nabla\zeta)\, dt + O(\tau),$$

where we have used the fact that $\Delta\zeta$ and $\nabla V \cdot \nabla\zeta$ are bounded functions, and used the shorthand $\rho_\tau(t) = \rho_\tau(t, \cdot)$, which is a probability measure on $\mathbb{R}^k$.

By using the total square distance estimate (here, with respect to the quadratic Wasserstein distance), we deduce that, for all fixed t_1 and t_2,

$$\int \rho_\tau(t_2)\zeta - \int \rho_\tau(t_1)\zeta + O(\tau) = \int_{t_1}^{t_2} \int \rho_\tau(t)(\Delta\zeta - \nabla V \cdot \nabla\zeta)\, dt + O(\tau).$$

If one then lets τ go to 0, this transforms into

$$\int \rho(t_2)\zeta - \int \rho(t_1)\zeta = \int_{t_1}^{t_2} \int \rho(t)(\Delta\zeta - \nabla V \cdot \nabla\zeta)\, dt,$$

which is the weak formulation of the linear Fokker-Planck equation with potential V,

$$\frac{\partial \rho}{\partial t} = \Delta\rho + \nabla \cdot (\rho \nabla V).$$

This equation should be satisfied by any weak cluster point of the sequence (ρ_τ); since it admits a unique solution, we deduce that this solution is the limit of the whole family ρ_τ as $\tau \to 0$. With this the argument is complete: *as the time-step goes to 0, the solution of the time-discretized gradient flow converges to the solution of the linear Fokker-Planck equation with initial datum* ρ_0. This is one way to justify the assertion that the Fokker-Planck equation really is the gradient flow for the free energy functional, when the distance is the quadratic Wasserstein distance.

Remark 8.12. In this case we have an explicit error bound $O(\tau)$, in some appropriate weak topology. In general it is an interesting problem to establish error estimates for such time-discretizations. As remarked in [**13**], error estimates can be established as soon as the functional under study is convex along the paths

$$t \longmapsto [T_1 + t(T_2 - T_1)]\#\mu,$$

for all μ, and all optimal transportation mappings T_1, T_2. Surprisingly, this condition holds true for all the displacement convex functionals which we have considered in Chapter 5.

8.5. Differentiability of the quadratic Wasserstein distance

We saw in the previous section a way to formalize the concept of gradient flow as the limit of a time-discretized problem. Recall, however, that in the Euclidean case we had a nice Euler-Lagrange equation, namely (8.18), at the level of the time-discretized problem. Is there anything similar in the present context?

It turns out that this is the case. In their study in progress, Ambrosio, Gigli and Savarè prove that, if F is a displacement convex functional, ρ_τ^n

the solution at n-th step of the time-discretized problem above, and $\nabla\varphi$ is an optimal transport between ρ_τ^n and $\rho = \rho_\tau^{n+1}$, then

$$\frac{\mathrm{Id} - \nabla\varphi^*}{\tau} \in -\partial^- F(\rho), \tag{8.30}$$

where ∂^- is a suitable notion of subdifferential. This is really a time-discretized version of the gradient flow, which in their formalism can be written

$$\dot\rho_t \in -\partial^- F(\rho_t),$$

with an appropriate definition of $\dot\rho_t$ living in the tangent space to $P_{\mathrm{ac},2}(\mathbb{R}^n)$ at ρ.

As a particular case, when F is differentiable at ρ and satisfies adequate smoothness assumptions, equation (8.30) becomes

$$\frac{\mathrm{Id} - \nabla\varphi^*}{\tau} = -\nabla\frac{\delta F}{\delta\rho}(\rho).$$

To establish these identities, one performs infinitesimal variations on the solution $\rho = \rho_\tau^{n+1}$, just as in the usual procedure to establish an Euler-Lagrange equation. In our context, the natural variations are once again those which are defined in terms of push-forward. So let ξ be a C^1, bounded vector field, and let $(T_t)_{-t_0<t<t_0}$ be the associated flow. We can define $(\rho_t)_{-t_0<t<t_0}$ in such a way that

$$\rho_t = T_t\#\rho,$$

and we know from Theorem 5.37 that it satisfies the continuity equation

$$\frac{\partial\rho_t}{\partial t} + \nabla\cdot(\xi\rho_t) = 0.$$

Since $\rho = \rho_0$ is a minimizer of

$$J(\rho) = \frac{W_2(\rho_\tau^n, \rho)^2}{2\tau} + F(\rho),$$

we know that $J(\rho_t) \geq J(\rho_0)$, so that the left derivative of $J(\rho_t)$ at $t = 0^-$ is nonpositive, and the right derivative at $t = 0^+$ is nonnegative. Under adequate regularity assumptions, one has

$$\left.\frac{d}{dt}\right|_{t=0} F(\rho_t) = -\int \frac{\delta F}{\delta\rho}\nabla\cdot(\rho\xi) = \int\left\langle\nabla\frac{\delta F}{\delta\rho}, \xi\right\rangle d\rho.$$

We shall show here that, under quite weak assumptions, the other part of $J(\rho_t)$, namely $W_2(\rho_\tau^n, \rho_t)^2/2\tau$, is differentiable at $t = 0$, and that its derivative is

$$\frac{1}{\tau}\int\langle x - \nabla\varphi^*(x), \xi\rangle\, d\rho(x),$$

where $\nabla\varphi^*\#\rho = \rho_\tau^n$. Since ξ is arbitrary, this will show that the gradient of $W_2(\rho_\tau^n, \rho_t)^2/2\tau$ is really $(\mathrm{Id} - \nabla\varphi^*)$; and this will make (8.30) plausible.

The differentiability of W_2^2 is expressed in the following theorem, which can be found in Ambrosio, Gigli and Savaré [**13**] in a slightly more general form.

Theorem 8.13 (Differentiability of the quadratic Wasserstein distance). *Let $\sigma \in P_{\mathrm{ac},2}(\mathbb{R}^n)$ be given, and let $G : \rho \longmapsto W_2(\sigma,\rho)^2$. Let $\rho_0 \in P_{\mathrm{ac},2}(\mathbb{R}^n)$, and let $(\rho_t)_{-t_0<t<t_0}$ be a path in $P_{\mathrm{ac},2}(\mathbb{R}^n)$ satisfying the regularity condition*

$$(\rho_t) \in \mathrm{AC}^2([0,1]; P_2(\mathbb{R}^n)),$$

and the continuity equation

$$\frac{\partial \rho_t}{\partial t} + \nabla \cdot (\rho_t \xi_t) = 0,$$

where $\xi_t(x)$ is a C^1 function of x and t, globally bounded. Then,

$$\begin{aligned}\frac{d}{dt}\bigg|_{t=0} G(\rho_t) &= 2\int \langle \nabla\varphi(x) - x,\, \xi \circ \nabla\varphi(x)\rangle\, d\sigma(x)\\ &= 2\int \langle y - \nabla\varphi^*(y),\, \xi(y)\rangle\, d\rho_0(y),\end{aligned}$$

where $\nabla\varphi$ is the optimal mapping between σ and ρ_0, for the transportation problem with quadratic cost function.

Remark 8.14. The regularity assumption means that for all $s,t \in (0,1)$, $W_2(\mu_s,\mu_t) \le \int_s^t \kappa(\tau)\,d\tau$, for some $\kappa \in L^2(d\tau)$. The result can be stated in the following compact way: for all fixed $\sigma \in P_{\mathrm{ac},2}(\mathbb{R}^n)$, the gradient of $W_2(\sigma,\cdot)^2/2$ at ρ_0 is the velocity field $\mathrm{Id} - \nabla\varphi^*$, which is the opposite of the optimal velocity field that one should use to transport ρ_0 back to σ. This theorem proves the remarkable fact that the quadratic Wasserstein distance is not only continuous with respect to the weak topology, but in fact *differentiable* for the natural differential structure with which it is associated.

Proof of Theorem 8.13. 1. Let T_t be the flow associated with the vector field ξ_t, in the sense that $(d/dt)T_t = \xi_t \circ T_t$. By Cauchy-Lipschitz theory, T_t defines a family of C^1 diffeomorphisms. From Theorem 5.34 we know that $T_t\#\rho_0 = \rho_t$, and as a consequence $T_t^{-1}\#\rho_t = \rho_0$. Moreover, the differential equation shows that $T_t - \mathrm{Id}$ and $T_t^{-1} - \mathrm{Id}$ are bounded uniformly for $|t| < t_0$, by $t_0\|\xi\|_{L^\infty}$. In particular, $W_2(\rho_0,\rho_t) \le \|\xi\|_{L^\infty} t$, which shows that ρ_t converges weakly to ρ_0 as $t \to 0$.

2. Let $\nabla\varphi$ be as in the statement of Theorem 8.13. Since $(T_t \circ \nabla\varphi)\#\sigma = \rho_t$, we can write

$$W_2(\sigma,\rho_t)^2 \le \int |x - T_t \circ \nabla\varphi(x)|^2\, d\sigma(x).$$

In particular,

$$\frac{W_2(\sigma,\rho_t)^2 - W_2(\sigma,\rho_0)^2}{t} \le \int \left\{ \frac{|x - T_t \circ \nabla\varphi(x)|^2 - |x - \nabla\varphi(x)|^2}{t} \right\} d\sigma(x).$$

By writing $|A|^2 - |B|^2 = \langle A-B, A+B\rangle$ and using the dominated convergence theorem to pass to the limit in the right-hand side, we obtain

$$\limsup_{t\downarrow 0} \frac{W_2(\sigma,\rho_t)^2 - W_2(\sigma,\rho_0)^2}{t} \le 2 \int \big\langle \nabla\varphi(x) - x,\, \xi \circ \nabla\varphi(x) \big\rangle\, d\sigma(x).$$

3. Let $\nabla\varphi_t$ be a gradient of a convex function such that $\nabla\varphi_t \# \sigma = \rho_t$. From the family $(\nabla\varphi_t)$ we extract a subsequence $(\nabla\varphi_{t_k})$, denoted $(\nabla\varphi_k)$ for brevity, such that

$$\limsup_{t\downarrow 0} \frac{W_2(\sigma,\rho_0)^2 - W_2(\sigma,\rho_t)^2}{t} = \lim_{k\to\infty} \frac{W_2(\sigma,\rho_0)^2 - W_2(\sigma,\rho_k)^2}{t_k}.$$

Then we write, reasoning as in Step 2,

$$\begin{aligned}
&\limsup_{t\downarrow 0} \frac{W_2(\sigma,\rho_0)^2 - W_2(\sigma,\rho_t)^2}{t} \\
&\le \limsup_{k\to\infty} \int \left\{ \frac{|x - T_{t_k}^{-1} \circ \nabla\varphi_k(x)|^2 - |x - \nabla\varphi_k(x)|^2}{t_k} \right\} d\sigma(x) \\
&= 2 \limsup_{k\to\infty} \int \left\langle \frac{(\mathrm{Id} - T_{t_k}^{-1}) \circ \nabla\varphi_k(x)}{t_k},\, x - \frac{(\mathrm{Id} + T_{t_k}^{-1}) \circ \nabla\varphi_k(x)}{2} \right\rangle d\sigma(x).
\end{aligned}$$

From our assumptions, ρ_k converges weakly to ρ_0 as $k \to \infty$. From Exercise 2.17, we know that $\nabla\varphi_k$ converges to $\nabla\varphi$ in measure, in the sense that

$$\forall \varepsilon > 0, \qquad \sigma[A_\varepsilon(k)] \xrightarrow[k\to\infty]{} 0,$$

where $A_\varepsilon(k) = \{x;\, |\nabla\varphi_k(x) - \nabla\varphi(x)| > \varepsilon\}$.

On the other hand, by using the bound $|T_t(x) - x| \le \|\xi\|_{L^\infty} t$ and the Cauchy-Schwarz inequality, we find that

$$\begin{aligned}
&2 \int_{A_\varepsilon(k)} \left\langle \frac{(\mathrm{Id} - T_{t_k}^{-1}) \circ \nabla\varphi_k(x)}{t_k},\, x - \frac{(\mathrm{Id} + T_{t_k}^{-1}) \circ \nabla\varphi_k(x)}{2} \right\rangle d\sigma(x) \\
&\le \|\xi\|_{L^\infty} \left(\int_{A_\varepsilon(k)} |x - \nabla\varphi_k(x)|\, d\sigma(x) + \|T_{t_k}^{-1} - \mathrm{Id}\,\|_{L^\infty} \right) \\
&\le \|\xi\|_{L^\infty} \sqrt{\int |x - \nabla\varphi_k(x)|^2\, d\sigma(x)}\, \sqrt{\sigma[A_\varepsilon(k)]} + \|\xi\|_{L^\infty}^2\, t_k \\
&= \|\xi\|_{L^\infty} W_2(\sigma,\rho_k) \sqrt{\sigma[A_\varepsilon(k)]} + \|\xi\|_{L^\infty}^2\, t_k,
\end{aligned}$$

which converges to 0 as $k \to \infty$, since $W_2(\sigma, \rho_k)$ is bounded and $\sigma[A_\varepsilon(k)] \to 0$.

Thus we may assume that $\nabla\varphi_k(x)$ is uniformly close to $\nabla\varphi(x)$. Similarly, the contribution of the set $|x| \geq R$ is negligible as $R \to \infty$, and we may assume that $|x|$ is bounded by R.

Then, since $\mathrm{Id} - T_t^{-1}$ is locally Lipschitz, we know that $(\mathrm{Id} - T_t^{-1})(\nabla\varphi_k(x))$ is very close to $(\mathrm{Id} - T_t^{-1})(\nabla\varphi(x))$ as soon as the inequalities $|\nabla\varphi_k(x) - \nabla\varphi(x)| \leq \varepsilon$ and $|x| \leq R$ are satisfied simultaneously. Putting together all this information, it is easy to establish that

$$\limsup_{k\to\infty} \int \left\langle \frac{(\mathrm{Id} - T_{t_k}^{-1}) \circ \nabla\varphi_k(x)}{t_k}, x - \frac{(\mathrm{Id} + T_{t_k}^{-1}) \circ \nabla\varphi_k(x)}{2} \right\rangle d\sigma(x)$$
$$\leq \int \langle \xi \circ \nabla\varphi(x), x - \nabla\varphi(x) \rangle \, d\sigma(x).$$

This establishes the right-derivability; by symmetry the left-derivability holds in a similar way.

We conclude that the function $t \longmapsto W_2(\sigma, \rho_t)^2$ is indeed differentiable at $t = 0$, and that its derivative is

$$2 \int \langle \nabla\varphi(x) - x, \xi \circ \nabla\varphi(x) \rangle \, d\sigma(x) = 2 \int \langle y - \nabla\varphi^*(y), \xi(y) \rangle \, d\rho_0(y).$$

This concludes the proof of Theorem 8.13. □

8.6. Non-quadratic costs

Even if the Benamou-Brenier formula was extended to rather general strictly convex cost functions by Brenier [**64**], the rest of the above discussion is not so easy to generalize. However, the case of homogeneous cost functions, $c(x - y) = |x - y|^p$, leads to a somewhat similar structure; for instance, it is possible to define geodesics. Also most of the discussion about time-discretization applies in this case. This delicate extension was considered by Otto in [**215**], in relation with the **p-Laplace equations**

$$\frac{\partial\rho}{\partial t} = \nabla \cdot (|\nabla\rho|^{p-2}\nabla\rho),$$

or again the porous medium equations

$$\frac{\partial\rho}{\partial t} = \Delta\rho^m.$$

Later the subject has been studied in depth in Agueh's PhD thesis [**3**]; in particular, a study of geodesics is performed in this reference. Note also the contribution by Cullen and Maroofi [**107**], where a non-quadratic cost function is used, in relation to the semi-geostrophic system.

Chapter 9

Entropy Production and Transportation Inequalities

In this chapter, we shall explore some links between optimal transportation, diffusive partial differential equations, and certain interesting classes of **functional inequalities**. The systematic study of this subject started around 1999 and has been rapidly evolving in the following years, involving increasingly vast areas of functional analysis. We shall not attempt to present an exhaustive survey, but rather try to give the reader an overview of this topic, together with a few selected proofs. More information can be found in lecture notes such as [**266, 265**], research papers such as [**46, 64, 79, 85, 97, 99, 219, 220**], or [**267**, Part II] and the many references therein.

As soon as partial differential equations are involved, many delicate arguments related to regularity issues should be taken care of at the level of the proofs. In order to limit the volume of these notes, we shall skip most of these problems and refer to the abovementioned papers for full justifications.

Our guideline in this chapter will be the general problem of studying the asymptotic behavior for linear or nonlinear diffusion equations. This is one of the major areas of applications for the functional inequalities introduced here. We strongly emphasize that this is not the only one, and that these inequalities have also been used for such issues as macroscopic limits of particle systems, explicit error estimates in numerical schemes or in problems of statistical inference, etc. Many references will be given later on.

From the preceding chapter we retain the following important principle: *some partial differential equations for probability distributions can be written as gradient flows of some energy functional $E(\rho)$ with respect to a differential structure inducing the quadratic Wasserstein distance W_2.* For many of those equations, the fact that they can be seen as gradient flows is not new. For instance, as we already mentioned, the heat equation can be seen as a gradient flow in $\dot{H}^{-1}(\mathbb{R}^n)$, or in $L^2(\mathbb{R}^n)$. But, as we shall see, the Wasserstein structure seems much better adapted to the study of certain topics. In particular, we shall see how it enables us to implement the natural guess that the trend to equilibrium for solutions of these partial differential equations should be linked to some displacement convexity properties of the energy.

Once again, we shall use the same notation ρ for a probability measure and for its Lebesgue density $d\rho/dx$.

9.1. More on optimal-transportation induced dissipative equations

We start this chapter with some heuristic considerations. Recall from the preceding chapter that we formally equip the space of absolutely continuous probability measures with the following metric tensor:

$$\left\| \frac{\partial \rho}{\partial s} \right\|_W^2 = \inf \left\{ \int \rho |v|^2; \qquad \frac{\partial \rho}{\partial s} + \nabla \cdot (\rho v) = 0 \right\}, \tag{9.1}$$

and that the gradient with respect to this metric structure is given by

$$\operatorname{grad}_W F(\rho) = -\nabla \cdot \left(\rho \nabla \frac{\delta F}{\delta \rho} \right), \tag{9.2}$$

where $\delta F/\delta\rho$ is nothing but the gradient of F with respect to the usual L^2 structure. So the corresponding gradient flow satisfies the equation

$$\frac{\partial \rho}{\partial t} = \nabla \cdot \left(\rho \nabla \frac{\delta F}{\delta \rho} \right). \tag{9.3}$$

As we explained in Chapter 8, for us this construction will remain formal, but it is possible to make sense of it.

In Chapter 5 we considered three basic kinds of energies:

- internal energy: $\mathcal{U}(\rho) = \displaystyle\int_{\mathbb{R}^n} U(\rho(x))\, dx;$
- potential energy: $\mathcal{V}(\rho) = \displaystyle\int_{\mathbb{R}^n} V\, d\rho = \int \rho(x) V(x)\, dx;$
- interaction energy: $\mathcal{W}(\rho) = \dfrac{1}{2} \displaystyle\int_{\mathbb{R}^n \times \mathbb{R}^n} W(x-y)\, d\rho(x)\, d\rho(y).$

If $F = \mathcal{U} + \mathcal{V} + \mathcal{W}$, then the corresponding gradient flow is

$$\frac{\partial \rho}{\partial t} = \nabla \cdot \Big[\rho \nabla U'(\rho) + \rho \nabla V + \rho(\rho * \nabla W)\Big], \qquad t \geq 0, \quad x \in \mathbb{R}^n. \tag{9.4}$$

Let us list once again a few examples of partial differential equations which fall into this category.

- $U(s) = s \log s$ (or equivalently $s \log s - s + 1$, which is nonnegative), $V = 0$, $W = 0$; then (9.4) is the heat equation.
- $U(s) = s^m/(m-1)$, $V = 0$, $W = 0$; then (9.4) is a porous-medium type equation.
- $U(s) = s \log s$, V is a given confinement potential; then (9.4) is the well-known linear Fokker-Planck equation.
- $U = 0$, $V = 0$, $W(z) = |z|^3/3$, $n = 1$; then (9.4) is the simple model for granular flow which was introduced by McNamara and Young [**207**] and later studied from a mathematical point of view by Benedetto, Caglioti and Pulvirenti [**41**].
- $U = 0$, $V(x) = -|x|^2/2$, $W(z) = |z|^3/3$, $n = 1$; then (9.4) is the rescaled version of this model which appears in the study of so-called homogeneous cooling states, see [**41**] or Problem 8 in Chapter 10.
- $U(s) = \sigma s \log s$, $V(x) = \alpha |x|^2/2$, $W(z) = |z|^3/3$ $n = 1$, $\sigma > 0$, $\alpha \geq 0$; then (9.4) is the equation studied by Benedetto, Caglioti, Carrillo and Pulvirenti [**40**] for the modelling of granular material undergoing diffusion in a heat bath.

There are many, many other instances of such equations, with various interaction potentials W, but the last three examples are particularly interesting for us in that they involve a convex W. This seems weird, since an interaction energy is supposed to decay as the distance becomes very large. But in fact, in these three last cases, the configuration space is a *velocity* phase space, and the corresponding equations are **kinetic equations**.

Another natural example of kinetic equation falling in the preceding category is the Fokker-Planck equation with quadratic potential $V(x) = |x|^2/2$, to be interpreted as a kinetic energy when the phase space is a velocity space.

As recalled in the last chapter, there are other examples of gradient flows, which are not necessarily of the form (9.4), but here we shall be content with these ones. In the sequel, we shall always act as if we were dealing with *smooth* solutions of these partial differential equations, decaying fast enough as $|x| \to \infty$, so that all the manipulations which we wish to

perform on them are allowed. In most situations, it is not very difficult to prove this fact under suitable assumptions on the initial datum: consult for instance [**219**] or the appendices in [**85**]. However, this is very tedious and requires some familiarity with the regularity theory for partial differential equations, which we do not want to necessarily require from the reader. Among the equations which we consider here, porous medium-type equations are the ones for which regularity theory is most tricky; a good source for this topic is Vázquez [**261**].

9.1.1. Entropy, entropy production and trend to equilibrium. In each of the cases considered above, $F(\rho)$ is a Lyapunov functional: whenever $(\rho_t)_{t\geq 0}$ is a solution, then $F(\rho_t)$ is a nonincreasing function of t. In fact this is a general property of gradient flows. By analogy with the entropy appearing in Boltzmann's H theorem, we shall call $-F$ an **entropy** of the system. Physically speaking, this is not so appropriate, since F should be called a **free energy** (energy − entropy×temperature). We shall assume that F admits a unique minimizer ρ_∞; recall from Section 5.3 that this uniqueness property holds true as soon as F is strictly displacement convex.

Let us denote by $D(\rho)$ the negative of the time-derivative of F along the flow. Namely,

$$D(\rho_t) = -\frac{d}{dt}F(\rho_t).$$

This functional D acts as an **entropy production**, or **dissipation of free energy**.

An explicit expression for D is easily established: at least when ρ_t enjoys sufficient regularity properties,

$$D(\rho) = \int_{\mathbb{R}^n} \rho \left|\nabla \frac{\delta F}{\delta \rho}\right|^2. \tag{9.5}$$

Exercise 9.1. Why could one have guessed this expression having in mind the Riemannian formalism of Chapter 8?

The problem of **trend to equilibrium in entropy sense** for equation (9.3) is to prove that

$$F(\rho_t) \xrightarrow[t\to\infty]{} F(\rho_\infty), \tag{9.6}$$

and if possible to obtain explicit estimates on the speed at which this convergence occurs, in terms of certain properties of the initial datum ρ_0.

In many cases of interest, convergence in the entropy sense implies convergence of ρ_t to ρ_∞ in the L^1 sense. For instance, in the case of the

Fokker-Planck equation,

$$F(\rho) - F(\rho_\infty) \geq \frac{1}{2}\|\rho - \rho_\infty\|_{L^1}^2.$$

This inequality is the so-called **Csiszár-Kullback-Pinsker inequality** [**104, 221**]. There are many variants of it, some of them studied in [**218, 266**].

A usually weaker notion of convergence is the **trend to equilibrium in the Wasserstein sense**: for instance, prove that

$$W_2(\rho_t, \rho_\infty) \xrightarrow[t\to\infty]{} 0. \tag{9.7}$$

As we saw at the end of Chapter 7, this kind of problem was first investigated in the seventies by Tanaka, in relation with the Boltzmann equation (which does not fall into our classification). Our choice of the quadratic Wasserstein distance is not innocent: precisely because it is the geodesic distance induced by Otto's differential point of view, it is often very well adapted to the study of these gradient flows.

When trying to prove convergence in the entropy sense, one can be lucky enough to prove an **entropy-entropy production inequality**, i.e. a functional inequality of the form

$$D(\rho) \geq \Phi\Big(F(\rho) - F(\rho_\infty)\Big), \tag{9.8}$$

where Φ is continuous and strictly increasing from 0. This inequality should be valid for all probability densities, or for a subclass which would be invariant under the flow associated to equation (9.3). Of course, since the functional D is the time-derivative of F along the flow, inequality (9.8) entails a differential inequality which implies convergence of $F(\rho_t)$ to $F(\rho_\infty)$ as $t \to \infty$. Moreover, there will be an explicit bound on the rate of convergence. For instance, if $\Phi(x) = Kx$, the entropy will approach its limit value exponentially fast; on the other hand, if $\Phi(x) = x^\alpha$ for some $\alpha > 1$, then the trend to equilibrium will be at least algebraic. This search for explicit rates of convergence will be one of our guidelines in this chapter; it should be noted that nonexplicit results are considerably easier to obtain.

9.1.2. Formal Hessian computations. We recall from Chapter 8 that $\mathrm{Hess}_W F$ (the Hessian of F with respect to the quadratic Wasserstein metric structure) can be computed via the formula

$$\left\langle \mathrm{Hess}_W F(\rho_0)\cdot \frac{\partial \rho_0}{\partial s}, \frac{\partial \rho_0}{\partial s}\right\rangle_W = \frac{d^2}{ds^2}\bigg|_{s=0} F(\rho_s), \tag{9.9}$$

where $\rho_s = \rho(s, \cdot)$ satisfies the geodesic equations (cf. Section 5.4)

$$(9.10) \qquad \begin{cases} \dfrac{\partial \rho}{\partial s} + \nabla \cdot (\rho v) = 0, \\[2ex] \dfrac{\partial(\rho v)}{\partial s} + \nabla \cdot (\rho v \otimes v) = 0, \end{cases} \qquad \begin{cases} \left.\dfrac{\partial \rho}{\partial s}\right|_{s=0} = \dfrac{\partial \rho_0}{\partial s}, \\[2ex] v_0 = \nabla u_0. \end{cases}$$

Note that here we use the notation s for the time parameter of the geodesic, in order to make sure that no confusion arises with the time t associated with the partial differential equations considered in this chapter.

If we denote $\xi = v_0$, then

$$(9.11) \quad \left\langle \mathrm{Hess}_W\, \mathcal{U}(\rho) \cdot \frac{\partial \rho}{\partial s}, \frac{\partial \rho}{\partial s} \right\rangle_W = \int_{\mathbb{R}^n} \left[P'(\rho)\rho - P(\rho)\right] (\nabla \cdot \xi)^2 + \int_{\mathbb{R}^n} P(\rho)\, \mathrm{tr}\, (\nabla \xi)^2,$$

where

$$P(\rho) = \int_0^\rho s U''(s)\, ds$$

is the **pressure** associated to the equation (9.3). Since ξ is a gradient, the matrix $\nabla \xi$ is symmetric, and therefore $\mathrm{tr}\, (\nabla \xi)^2$ coincides with the square of the Hilbert-Schmidt norm of $\nabla \xi$.

The Hessian of the potential energy is easier to compute:

$$(9.12) \qquad \left\langle \mathrm{Hess}_W\, \mathcal{V}(\rho) \cdot \frac{\partial \rho}{\partial s}, \frac{\partial \rho}{\partial s} \right\rangle_W = \int_{\mathbb{R}^n} \langle D^2 V \cdot \xi, \xi \rangle \, d\rho.$$

The Hessian of the interaction energy needs more work:

$$(9.13) \quad \left\langle \mathrm{Hess}_W\, \mathcal{W}(\rho) \cdot \frac{\partial \rho}{\partial s}, \frac{\partial \rho}{\partial s} \right\rangle_W$$
$$= \frac{1}{2} \int_{\mathbb{R}^n \times \mathbb{R}^n} \Big\langle D^2 W(x-y) \cdot [\xi(x) - \xi(y)], [\xi(x) - \xi(y)] \Big\rangle \, d\rho(x)\, d\rho(y).$$

Extracting some strict positivity from these expressions will be a key ingredient in the sequel.

At this point we should point out one of the main advantages in the differential formalism developed in Chapter 8: with its help, one can easily perform tricky computations, from a formal point of view. Here is one example: it is often very useful to have access to the second derivative (with respect to time) of the entropy functional along equation (9.3). For the general equation (9.4), this is an incredibly tedious task to perform! But if we think of it as a gradient flow, then things become much easier, by means of the abstract identity

$$\frac{d}{dt} \| \mathrm{grad}\, F(\rho_t) \|_W^2 = -2 \Big\langle \mathrm{Hess}_W\, F(\rho_t) \cdot \mathrm{grad}_W\, F(\rho_t), \mathrm{grad}_W\, F(\rho_t) \Big\rangle_W.$$

As an exercise for the reader, he or she can write down the explicit form of $d^2F(\rho_t)/dt^2$ for all the equations which we have encountered, with almost no calculation. What is more important, the result comes in a very appealing form: for instance, in the second derivative of $\mathcal{U} + \mathcal{V} + \mathcal{W}$, one directly obtains three terms which correspond respectively to the action of the three Hessian operators. Practically, arriving at such an expression without any guideline would not be a feasible task.

Of course, our construction in Chapter 8 stayed on a formal level, so the reader may feel embarrassed to use it in such calculations. But even the possibility to have just a guess of the correct expression, by these formal means, is particularly useful. Once the final result has been guessed, it is much easier to establish it directly. Moreover, the procedure can be justified under a few regularity assumptions, which we do not discuss here.

9.1.3. Basic principles. Up to this date, three basic principles have been identified, which relate entropy, entropy dissipation and Wasserstein distance; they are not always true, but we shall give simple sufficient conditions for them in the sequel. We shall use the convenient notation

$$F(\rho|\rho_\infty) = F(\rho) - F(\rho_\infty).$$

Rule # 1: $D(\rho)$ dominates $F(\rho|\rho_\infty)$.

Rule # 2: $F(\rho|\rho_\infty)$ dominates $W_2(\rho,\rho_\infty)$.

Rule # 3: A little bit of $D(\rho)$ and $W_2(\rho,\rho_\infty)$ together dominate $F(\rho|\rho_\infty)$.

Here is the basic abstract theorem on the subject [**219**]. In the present notes, we shall consider it as a formal theorem, which in fact should be checked explicitly on each particular example of interest. Most of the time, a rigorous proof can be obtained by mimicking the abstract proof. It is quite likely that a rigorous version of this theorem, possibly with some additional regularity assumptions, could be established by means of the construction in [**13, 85**].

Formal Theorem 9.2 (Functional inequalities associated with displacement convexity). *Let F be a lower semi-continuous functional on $P(\mathbb{R}^n)$, equipped with the distance W_2. Then,*

(i) If F is uniformly displacement convex with constant $\lambda > 0$, then it admits a unique minimizer ρ_∞ in $P_2(\mathbb{R}^n)$, and

$$\forall \rho \in P_2(\mathbb{R}^n), \qquad D(\rho) \geq 2\lambda F(\rho|\rho_\infty), \tag{9.14}$$

where $F(\rho|\rho_\infty) \equiv F(\rho) - F(\rho_\infty)$.

(ii) If (9.14) *holds, then*

$$\forall \rho \in P_2(\mathbb{R}^n), \qquad W_2(\rho, \rho_\infty) \le \sqrt{\frac{2}{\lambda} F(\rho|\rho_\infty)}. \tag{9.15}$$

(iii) If F is displacement convex, then

$$\forall \rho \in P_2(\mathbb{R}^n), \qquad F(\rho|\rho_\infty) \le W_2(\rho, \rho_\infty)\sqrt{D(\rho)}. \tag{9.16}$$

More generally, if $\mathrm{Hess}_W F \ge -C\,\mathrm{Id}$, *with* $C \in \mathbb{R}$, *then*

$$\forall \rho \in P_2(\mathbb{R}^n), \qquad F(\rho|\rho_\infty) \le W_2(\rho, \rho_\infty)\sqrt{D(\rho)} + \frac{C}{2} W_2(\rho, \rho_\infty)^2. \tag{9.17}$$

Formal Corollary 9.3 (Displacement convexity and trend to equilibrium). *If F is λ-uniformly displacement convex for some $\lambda > 0$ then one has for equation* (9.3) *the following estimates of trend to equilibrium,*

$$F(\rho_t|\rho_\infty) \le F(\rho_0|\rho_\infty) e^{-2\lambda t}, \tag{9.18}$$

$$W_2(\rho_t, \rho_\infty) \le \sqrt{\frac{2}{\lambda}\, F(\rho_0|\rho_\infty)}\, e^{-\lambda t}. \tag{9.19}$$

Remarks 9.4. (i) The estimate $W_2(\rho_t, \rho_\infty) \le W_2(\rho_0, \rho_\infty) e^{-\lambda t}$, which is (in view of (9.15)) stronger than (9.19), can also be proven, see [**85**].

(ii) Corollary 9.3 is a statement of trend to equilibrium in the entropy sense or in the Wasserstein sense, only when the energy at the initial time is finite. The latter assumption can however be dispensed with when one is dealing with a displacement convex energy. Indeed, in this case one can obtain the abstract estimate

$$F(\rho_t) \le \frac{W_2(\rho_0, \rho_\infty)^2}{4t}. \tag{9.20}$$

When it applies, the energy becomes finite for any positive time, whatever the initial datum in $P_2(\mathbb{R}^n)$. In fact, one even has

$$D(\rho_t) \le \frac{W_2(\rho_0, \rho_\infty)^2}{t^2}.$$

In the case of the heat or porous-medium equations, this can be interpreted as a statement of **parabolic regularization** (see for instance Evans [**133**] for an introduction to parabolic regularity). However, we emphasize that it is a general feature of gradient flows for a convex energy. Consult [**220**] for an abstract scheme of proof and a rigorous implementation in the case $F(\rho) = \int \rho \log \rho + \int V\, d\rho$, V convex, where the gradient flow is the linear Fokker-Planck equation.

9.1.4. Abstract proofs. We now present proofs for Theorem 9.2, following [**219**, Section 3]. Since this will remain a formal theorem, we shall be a little bit sketchy.

"Proof" of (i) (Bakry-Émery strategy). This strategy consists in the idea of comparing the first and second derivatives of $F(\rho_t)$, rather than directly comparing F with its first derivative. So let us introduce the dissipation of the dissipation of F, namely

$$DD(\rho) = 2\left\langle \mathrm{Hess}_W F(\rho)\cdot\xi,\xi\right\rangle_W,$$

where $\xi = -\,\mathrm{grad}_W F(\rho)$. From our convexity assumption, we see that

$$DD(\rho) \geq 2\lambda\|\xi\|_W^2 = 2\lambda D(\rho).$$

As a consequence, for all $t > 0$, $DD(\rho_t) - 2\lambda D(\rho_t) \geq 0$. In particular, $D(\rho_t) \longrightarrow 0$ exponentially fast.

Now, let $\phi(t) = D(\rho_t) - \lambda F(\rho_t|\rho_\infty)$. Of course

$$\phi'(t) = -[DD(\rho_t) - \lambda D(\rho_t)] \leq 0.$$

Taking for granted that $F(\rho_t) \to F(\rho_\infty)$ as $t \to \infty$ (without assuming any rate), we find that ϕ is a nonincreasing function going to 0 as $t \to \infty$. Hence $\phi(0) \geq 0$, which was our goal. □

Remarks 9.5. (i) As we mentioned above, the Bakry-Émery strategy consists in the idea of comparing the first to the second derivative of the entropy functional, and then of "integrating in time". One of the main merits of Bakry and Émery was to devise a series of calculus rules which allowed them to perform this second derivative computation. The "Γ_2 formalism" developed by Bakry and Émery in [**28**] and later papers to achieve this goal has however very little to do with our mass transportation setting.

(ii) Part (i) of Theorem 9.2 can also be deduced from part (iii) with $C = -\lambda$, by means of the elementary inequality

$$W_2\sqrt{D} \leq \frac{\lambda W_2^2}{2} + \frac{D}{2\lambda}.$$

This line of reasoning has the great merit of avoiding the use of partial differential equations (the gradient flow) in the proof, and is easier to implement. In the sequel, we shall illustrate this on the case of the linear Fokker-Planck equation.

(iii) A generalization of the Bakry-Émery method is presented in Problem 11 of Chapter 10.

"Proof" of (ii). We want to bound from above the time-increase of the Wasserstein distance $W_2(\rho_0, \rho_t)$. So let us consider the upper right derivative

$$\frac{d}{dt}\bigg|^+ W_2(\rho_0, \rho_t) = \limsup_{h \downarrow 0} \frac{W_2(\rho_0, \rho_{t+h}) - W_2(\rho_0, \rho_t)}{h}.$$

Using the triangle inequality for W_2, we see that

$$\frac{d}{dt}\bigg|^+ W_2(\rho_0, \rho_t) \leq \limsup_{h \downarrow 0} \frac{W_2(\rho_t, \rho_{t+h})}{h}.$$

By definition of Riemannian structure, this last quantity is formally equal to

$$\left\|\frac{\partial \rho_t}{\partial t}\right\|_W = \|\mathrm{grad}_W F(\rho_t)\|_W = \sqrt{D(\rho_t)}.$$

But, as a consequence of (9.14),

$$\sqrt{D(\rho_t)} \leq \frac{D(\rho_t)}{\sqrt{2\lambda F(\rho_t|\rho_\infty)}};$$

so, on the whole,

$$\frac{d}{dt}\bigg|^+ W_2(\rho_0, \rho_t) \leq \sqrt{D(\rho_t)} \leq \frac{D(\rho_t)}{\sqrt{2\lambda F(\rho_t|\rho_\infty)}}. \tag{9.21}$$

Let

$$\phi(t) = \sqrt{\frac{2}{\lambda} F(\rho_t|\rho_\infty)} + W_2(\rho_0, \rho_t).$$

From (9.21) we see that

$$\frac{d}{dt}\bigg|^+ \phi(t) = -\frac{D(\rho_t)}{\sqrt{2\lambda F(\rho_t|\rho_\infty)}} + \frac{d}{dt}\bigg|^+ W_2(\rho_0, \rho_t) \leq 0,$$

which means that ϕ is a nonincreasing function of t. Taking for granted that $\phi(t) \to W_2(\rho_0, \rho_\infty)$ as $t \to \infty$, it follows that

$$W_2(\rho_0, \rho_\infty) = \lim_{t \to \infty} \phi(t) \leq \phi(0) = \sqrt{\frac{2F(\rho_0|\rho_\infty)}{\lambda}}.$$

□

"Proof" of (iii). We shall directly prove (9.17). Fix an arbitrary $\rho \in P(\mathbb{R}^n)$ and consider the geodesic $(\rho_s)_{0 \leq s \leq 1}$ joining $\rho_0 = \rho$ to $\rho_1 = \rho_\infty$.

First of all, $\|\partial \rho_s / \partial s\|_W$ is constant and thus equal to $W_2(\rho_0, \rho_1)$. This is a general rule for geodesics in Riemannian geometry.

Then, write the Taylor formula

$$F(\rho_1) = F(\rho_0) + \left\langle \mathrm{grad}_W F(\rho_0), \left.\frac{\partial \rho_s}{\partial s}\right|_{s=0} \right\rangle_W + \int_0^1 (1-s) \left\langle \mathrm{Hess}_W F(\rho_s) \cdot \frac{\partial \rho_s}{\partial s}, \frac{\partial \rho_s}{\partial s} \right\rangle_W ds. \tag{9.22}$$

On one hand, one can bound the scalar product,

$$\left\langle \mathrm{grad}_W F(\rho_0), \left.\frac{\partial \rho_s}{\partial s}\right|_{s=0} \right\rangle_W \leq \|\mathrm{grad}_W F(\rho_0)\|_W \left\| \left.\frac{\partial \rho_s}{\partial s}\right|_{s=0} \right\|_W = \sqrt{D(\rho_0)}\, W_2(\rho_0, \rho_1);$$

on the other hand, by assumption

$$\left\langle \mathrm{Hess}_W F(\rho_s) \cdot \frac{\partial \rho_s}{\partial s}, \frac{\partial \rho_s}{\partial s} \right\rangle_W ds \geq -C \left\| \frac{\partial \rho_s}{\partial s} \right\|_W^2 = -C\, W_2(\rho_0, \rho_1)^2.$$

By plugging these two inequalities into (9.22), we get the desired result. □

Remark 9.6. Writing the Taylor formula

$$F(\rho_0) = F(\rho_1) - \left\langle \mathrm{grad}_W F(\rho_1), \left.\frac{\partial \rho_s}{\partial s}\right|_{s=1} \right\rangle_W + \int_0^1 s \left\langle \mathrm{Hess}_W F(\rho_s) \cdot \frac{\partial \rho_s}{\partial s}, \frac{\partial \rho_s}{\partial s} \right\rangle_W ds,$$

one also gets a direct "proof" that (9.15) holds true when F is λ-uniformly displacement convex.

9.1.5. A particular, important case.

All the preceding considerations are rather abstract, so we shall now consider "concrete" examples. First, in the next three sections we shall analyze in some detail the case when $U(s) = s \log s$ and $W = 0$ (no interaction), so that equation (9.3) is the **linear Fokker-Planck equation**

$$\frac{\partial \rho}{\partial t} = \Delta \rho + \nabla \cdot (\rho \nabla V) = \nabla \cdot \big[\rho \nabla (\log \rho + V)\big]. \tag{9.23}$$

This linear drift-diffusion equation has considerable importance in many domains of mathematics and physics. A quite unusual aspect of our study is that we have rewritten this simple, linear diffusive equation as a nonlinear, complicated transport equation. Strange as it may seem, this point of view will uncover several interesting features of the equation. Actually, this point of view has also been used for a long time in numerical simulations, where it is called the method of "diffusion velocity". Remarkably enough, it was also developed independently by Voiculescu for his theory of free probability

(references can be found in [**42**], where the authors adapt the proof of Theorem 9.2 (ii) to the setting of free probability, with the Gaussian distribution replaced by Wigner's semi-circular density).

Without loss of generality, we shall assume that

$$\int_{\mathbb{R}^n} e^{-V} = 1,$$

and define $\rho_\infty = e^{-V}$. In the case of the Fokker-Planck equation, the general abstract objects which we introduced coincide with well-known functionals:

$$F(\rho|\rho_\infty) = \int \rho \log \frac{\rho}{\rho_\infty} \equiv H(\rho|\rho_\infty) \tag{9.24}$$

is known as the free energy functional (up to a nonessential additive constant), but also as the **relative Kullback information** of ρ with respect to ρ_∞. Next,

$$D(\rho) = \int \rho \left|\nabla \left(\log \frac{\rho}{\rho_\infty}\right)\right|^2 \equiv I(\rho|\rho_\infty) \tag{9.25}$$

is the **relative Fisher information** of ρ with respect to ρ_∞.

Exercise 9.7 (Kullback information). Show that the Kullback information $H(\rho|\rho_\infty)$ is always nonnegative.

Hint: When f is a probability density with respect to a probability measure μ, then $\int f \log f \, d\mu = \int (f \log f - f + 1)\, d\mu$.

Exercise 9.8. Show that both the relative Kullback information and the relative Fisher information are convex functionals of ρ (in the usual sense).

Hint: Use the convexity of $(\rho, m) \longmapsto |m|^2/\rho$ on $\mathbb{R}_+ \times \mathbb{R}^n$.

Actually, both the relative Fisher information and the relative Kullback information are always well-defined in $[0, +\infty]$, with the convention that they are infinite if ρ is not absolutely continuous with respect to ρ_∞. Note that the definition of I requires a differential structure on the basis space (in our case $\mathbb{R}^n$), contrary to that of H.

The Kullback information is a basic object in information theory, as developed after the work of Shannon. It is obviously reminiscent of Boltzmann's H functional; the only difference is the reference measure, which is ρ_∞ in the case of $H(\rho|\rho_\infty)$, and just Lebesgue measure in the case of the H functional.

The relative Fisher information is also often called a "Dirichlet form" because it can be rewritten as $4 \int |\nabla \sqrt{h}|^2 \rho_\infty$, if one sets $h = \rho/\rho_\infty$. Strictly

speaking, the Fisher information is the functional

$$(9.26)\quad I(\rho) = \int \rho \big|\nabla \log \rho\big|^2 = \int \frac{|\nabla \rho|^2}{\rho} = 4 \int |\nabla \sqrt{\rho}|^2 = -\int \rho\, \Delta(\log \rho),$$

all these identities being equivalent under suitable regularity assumptions. So the relation between the Fisher information and the relative Fisher information, is just the same as between the H functional and the relative Kullback information.

The functional I was introduced by Fisher [**143**] as part of his theory of "efficient statistics", and since then it has become a key tool in information theory and statistics. For these topics the reader may consult Cover and Thomas [**102**] and the huge list of references therein.

In this context, inequalities such as (9.14) have been studied for a long time, under the name of **logarithmic Sobolev inequalities**. As for inequalities of the type (9.15), they are sometimes referred to as **transportation inequalities**. However, since this book is full of transportation inequalities, we shall rather call them **Talagrand inequalities**, because they were first established by Talagrand [**247**] for Gaussian reference measures. As for (9.16), they were called **HWI inequalities** because they involve at the same time the relative information H, the Wasserstein distance W_2 and the Fisher information I.

The next three sections will be devoted to a brief review of these inequalities, which by now (together with their variants) constitute a domain of functional analysis by themselves.

9.2. Logarithmic Sobolev inequalities

Let $\rho_\infty = e^{-V}$ be a reference probability distribution on $\mathbb{R}^n$, absolutely continuous with respect to Lebesgue measure. As we said earlier, we agree that ρ_∞ satisfies a logarithmic Sobolev inequality with constant $\lambda > 0$ (for short: **LSI**(λ)) if

$$(9.27)\qquad \forall \rho \in P(\mathbb{R}^n), \qquad H(\rho|\rho_\infty) \le \frac{1}{2\lambda} I(\rho|\rho_\infty).$$

Some probability measures ρ_∞ satisfy a logarithmic Sobolev inequality, and some do not. In dimension 1, a necessary and sufficient condition for a logarithmic Sobolev inequality has been established by Bobkov and Götze [**48**]:

$$\sup_{x \ge m} \left(\int_x^{+\infty} \rho_\infty(t)\, dt \right) \left(\int_m^x \frac{dt}{\rho_\infty(t)} \right) \left(\log \frac{1}{\int_x^{+\infty} \rho_\infty(t)\, dt} \right) < +\infty,$$

$$\sup_{x \leq m} \left(\int_{-\infty}^{x} \rho_\infty(t)\, dt \right) \left(\int_{x}^{m} \frac{dt}{\rho_\infty(t)} \right) \left(\log \frac{1}{\int_{-\infty}^{x} \rho_\infty(t)\, dt} \right) < +\infty,$$

where m is a median of ρ_∞, i.e. a real number satisfying

$$\int_{-\infty}^{m} d\rho_\infty = \int_{m}^{+\infty} d\rho_\infty = \frac{1}{2}.$$

From this criterion the reader may check, as an exercise, that if $V(x)$ behaves as $|x|^\alpha$ for $|x| \to \infty$, then the probability measure $e^{-V(x)}$ satisfies a logarithmic Sobolev inequality if and only if $\alpha \geq 2$. Thus, a logarithmic Sobolev inequality implies a *very fast decay* of the probability measure as $|x| \to \infty$.

However, this one-dimensional necessary and sufficient condition is not always easy to check. In dimension greater than 1, no such characterization is known, and it has been an important point to find some simple criteria ensuring that a probability measure satisfies a logarithmic Sobolev inequality. Some of the most important results in the subject are summarized in the next theorem.

Theorem 9.9 (Necessary or sufficient conditions for logarithmic Sobolev inequalities). *(i) Let ρ_1 and ρ_2 be two probability measures on $\mathbb{R}^{n_1}$ and $\mathbb{R}^{n_2}$ respectively, satisfying logarithmic Sobolev inequalities LSI(λ). Then also $\rho_1 \otimes \rho_2$ satisfies LSI(λ) on $\mathbb{R}^{n_1+n_2}$.*

(ii) Let $\widetilde{\rho}_\infty = e^{-v}\rho_\infty$, where ρ_∞ satisfies LSI(λ) and v is bounded. Then $\widetilde{\rho_\infty}$ satisfies LSI(λ'), where

$$\lambda' = \exp(-2\,\mathrm{osc}\,(v))\lambda, \qquad \mathrm{osc}\,(v) = \sup v - \inf v.$$

(iii) Let V be a C^2 function on $\mathbb{R}^n$ with $\int e^{-V} = 1$ and $D^2V \geq \lambda I_n$ for some $\lambda > 0$. Then e^{-V} satisfies LSI(λ).

(iv) If ρ_∞ satisfies a logarithmic Sobolev inequality on $\mathbb{R}^n$, then there exists $\alpha > 0$ such that

$$\int e^{\alpha|x|^2}\, d\rho_\infty(x) < +\infty. \tag{9.28}$$

Part (i) of this theorem is easy and can be treated as an exercise. The interesting point is that the constants are preserved under tensorization. Part (ii) is not very difficult; it is due to Holley and Stroock [**168**]. Part (iii), on the other hand, is a celebrated and deep result (actually slightly older than part (ii)) known as the **Bakry-Émery theorem** [**28**]. The combination of (ii) and (iii) applies to many interesting potentials in statistical physics, such as the double-well potential $V(x) = ax^4 - bx^2$. Finally, part (iv) is due to Herbst, and its proof can be found in [**181**], for instance. The notable

contribution by Djellout, Guillin and Wu [**120**] considered the necessary condition (9.28) in the equivalent formulation

$$\exists \alpha > 0; \qquad \int e^{\alpha|x-y|^2}\, d\mu(x)\, d\mu(y) < +\infty, \tag{9.29}$$

which is also equivalent to (9.45) below. These inequalities will turn out to be weaker than a logarithmic Sobolev inequality.

Bakry and Émery's theorem was proven around 1985; the original proof was rewritten and improved by Ledoux [**179**] at the beginning of the nineties. But it was only with [**219**] that some links between this theorem and mass transportation were established. As we shall see in Section 9.4, the Bakry-Émery theorem can be established in a very simple way via optimal transportation; in fact the reader already has all the elements to do so! But first of all, let us give some background on logarithmic Sobolev inequalities and explain why they are interesting.

9.2.1. Background. First of all, we mention two surveys about logarithmic Sobolev inequalities. The first is due to Gross [**161**]; it was exhaustive in 1992, but the literature on the subject has blown up since then. The second source which we recommend to the reader is the book (written in French) by the research group in Toulouse [**17**].

The archetypal logarithmic Sobolev inequality is the Gaussian one. Let $\gamma(x) = \gamma_n(x) = e^{-|x|^2/2}/(2\pi)^{n/2}$ denote the standard centered Gaussian distribution with identity covariance matrix on $\mathbb{R}^n$; then it satisfies LSI(1). In other words,

$$\forall \rho \in P(\mathbb{R}^n), \qquad H(\rho|\gamma) \leq \frac{1}{2} I(\rho|\gamma). \tag{9.30}$$

This particular inequality has a long history. Actually it was proven by Stam [**241**] in 1959 in the equivalent formulation

$$\forall \rho \in P(\mathbb{R}^n) \qquad \mathcal{N}(\rho) I(\rho) \geq n, \tag{9.31}$$

where $\mathcal{N}$ stands for Shannon's entropy power functional, defined in (6.13).

Stam however proved inequality (9.31) only in dimension $n = 1$, and did not realize all its interest. Several years later, Federbush established the same inequality by different means; but it is only with the work of Gross in 1975 that the logarithmic Sobolev inequality for the Gaussian measure became famous. He proved the other equivalent formulation

(9.32) $\forall h \in L^2(d\gamma)$,

$$\int |h|^2 \log |h|^2\, d\gamma \leq 2 \int |\nabla h|^2\, d\gamma + \left(\int |h|^2\, d\gamma\right) \log \left(\int |h|^2\, d\gamma\right).$$

Meanwhile, Stam's contribution had been forgotten, and was rediscovered only in the early nineties [**83**]. Accordingly, we shall refer to the equivalent inequalities (9.30), (9.31) and (9.32) as the **Stam-Gross logarithmic Sobolev inequality**.

One reason logarithmic Sobolev inequalities are very famous is that they constitute a crossing point of several families of functional inequalities:

• *Sobolev-type inequalities in large dimension.* Classical Sobolev inequalities assert the embedding of the Sobolev space $W^{1,p}(\mathbb{R}^n)$ into $L^{p^\star}(\mathbb{R}^n)$ for $p^\star = (np)/(n-p)$ whenever $p < n$; recall Section 6.4 in Chapter 6. There are many other Sobolev inequalities, for instance $W^{1,p}(\mathbb{R}^n) \subset C^{0,\alpha}(\mathbb{R}^n)$ for $n > p$, or variants for the spaces $W^{k,p}(\mathbb{R}^n)$, made up of functions whose derivatives up to order k all lie in $L^p(\mathbb{R}^n)$. A classical reference for all this is Adams [**2**]. In the sequel we shall use the common notation $H^1 = W^{1,2}$, which recalls that $W^{1,2}$ is a Hilbert space.

From the formulation (9.30) the denotation of "logarithmic Sobolev inequality" can be easily understood. This inequality asserts the embedding of the weighted Sobolev space

$$H^1(d\gamma) = \{h \in L^2(d\gamma); \quad \nabla h \in L^2(d\gamma)\}$$

into the Orlicz space

$$L^2 \log L(d\gamma) = \left\{h \in L^2(d\gamma); \quad \int |h|^2 \log|h| \, d\gamma < +\infty\right\}.$$

At first sight this seems like a weaker statement than plain Sobolev embedding. However, it is stronger in several respects. First, it is taken with respect to the Gaussian measure (the standard Sobolev embedding does not hold in this case). Secondly, both the embedding space and the constant of embedding are **independent of the dimension**. Compare with, for instance, the Sobolev embedding $H^1(\mathbb{R}^n) \subset L^{\frac{2n}{n-2}}(\mathbb{R}^n)$ $(n \geq 3)$. On one hand, the exponent $(2n)/(n-2)$ becomes closer and closer to 2 as $n \to \infty$; on the other hand, the constant of embedding blows up in the same limit. In fact, (9.32) is all that remains of the Sobolev embedding when one is interested in an embedding which would not depend on the dimension. This statement can be made quite precise: Beckner [**36**] has shown how inequality (9.32) can be obtained from the Sobolev embedding $H^1(\mathbb{R}^N) \subset L^{\frac{2N}{N-2}}(\mathbb{R}^N)$ *with sharp constants*, letting the dimension N go to infinity. More precisely, he uses a variant of this optimal Sobolev inequality set on the N-dimensional sphere $S^N(\sqrt{N})$ with radius $\sqrt{N}$ (the Sobolev inequality for H^1 is conformally invariant, so it can be rewritten on the sphere after use of a conformal mapping). Then, to recover the logarithmic Sobolev

inequality for the Gaussian measure, he uses the fact that

$$\gamma_n = \lim_{N\to\infty} P_{N,n}\#\sigma_N,$$

where γ_n stands for the Gaussian measure on $\mathbb{R}^n$, σ_N for the uniform measure on $S^N(\sqrt{N})$, and $P_{N,n}$ for the orthogonal projection from $\mathbb{R}^{N+1}$ to $\mathbb{R}^n$.

Thus, in some sense logarithmic Sobolev inequalities are the replacement for Sobolev inequalities in infinite dimension, and as such they have become a very useful tool in many fields. This is not surprising, since Sobolev inequalities are among the most universal tools in many branches of functional analysis. The original motivation of Federbush and Gross was mathematical **quantum field theory**, a topic which was developed in the seventies by the efforts of Glimm, Jaffe, Nelson and many others; see [**156**] for an introduction.

Later it was understood that this infinite-dimensional character also made logarithmic Sobolev inequalities a precious tool in the study of trend to equilibrium for spin systems (see for instance the survey by Guionnet and Zegarlinski [**162**]), or hydrodynamic scalings for systems of interacting particles (see the survey by Yau [**269**]). Among the important problems in these fields is the possibility of establishing logarithmic-type Sobolev inequalities adapted to the systems under study [**184, 268**].

• *Poincaré-type inequalities.* It is known since the work of Rothaus [**230**] that the inequality LSI(λ) implies the **Poincaré inequality P(λ)**:
(9.33)

$$\forall f \in L^2(d\rho_\infty), \qquad \left[\int f\, d\rho_\infty = 0 \Longrightarrow \int f^2\, d\rho_\infty \le \frac{1}{\lambda}\int |\nabla f|^2\, d\rho_\infty\right].$$

This is nothing but an estimate on the spectral gap for the linear operator $L = -\Delta + \nabla V\cdot\nabla$, where $V = -\log\rho_\infty$. Note that (9.33) is equivalent to (8.13). One of the reasons why L is an important object is that it represents the Dirichlet form for ρ_∞ in the sense that

$$\int (Lf)\, g\, d\rho_\infty = \int f\,(Lg)\, d\rho_\infty = \int \nabla f\cdot\nabla g\, d\rho_\infty.$$

The connection between (9.27) and (9.33) can be seen very simply by a linearization procedure, in the same way as in Section 7.6: if we set $\rho = (1+\varepsilon f)\rho_\infty$, then as $\varepsilon\to 0$,

$$H(\rho|\rho_\infty) \simeq \frac{\varepsilon^2}{2}\int f^2\, d\rho_\infty, \qquad I(\rho|\rho_\infty) = \varepsilon^2\int |\nabla f|^2\, d\rho_\infty.$$

Thus (9.27) turns into (9.33) in the limit as $\varepsilon\to 0$.

• *Isoperimetric-type inequalities.* As we already explained in Chapter 6, the Sobolev embedding $W^{1,1}(\mathbb{R}^n)\subset L^{n/(n-1)}(\mathbb{R}^n)$, with optimal constants, is a

reformulation of the standard isoperimetric inequality in $\mathbb{R}^n$. For a comprehensive account on known connections between Sobolev and isoperimetric inequalities, one may consult the monograph by Bobkov and Houdré [**49**]. Beckner [**35**] noticed that the Stam-Gross inequality can be seen as a limit case of **Bobkov's Gaussian isoperimetric inequality** [**47**], also by a linearization procedure. Just to give the reader a tiny flavor of what goes on in this field, let us recall the basic facts about this Gaussian isoperimetric inequality.

Whenever B is a measurable set in $\mathbb{R}^n$, call $\gamma[B]$ its Gaussian volume, and define its Gaussian surface by

$$S_\gamma(B) = \liminf_{t\downarrow 0} \frac{\gamma(B_t) - \gamma(B)}{t},$$

where B_t is obtained from B by thickening in Euclidean distance,

$$B_t = \big\{x \in \mathbb{R}^n; \quad d(x,B) \le t\big\}, \qquad d(x,B) = \inf_{y\in B} \|x-y\|_{\mathbb{R}^n}.$$

Then the Gaussian isoperimetry [**54, 243**] states that, for fixed Gaussian volume, half-spaces have maximal Gaussian surface. Bobkov's functional version of this statement can be stated as follows. Let $\mathcal{U}$ denote the Gaussian isoperimetric function, i.e. $\mathcal{U} = \gamma_1 \circ \Gamma^{-1}$, where $\gamma_1(x) = (2\pi)^{-1/2} e^{-|x|^2/2}$ is the one-dimensional Gaussian measure, and $\Gamma(x) = \int_{-\infty}^x \gamma_1(s)\,ds$. A few moments of reflection show that $\mathcal{U}(x)$ is the Gaussian surface of the half-space with Gaussian volume x. Then, for all functions $h : \mathbb{R}^n \to [0,1]$,

$$\mathcal{U}\left(\int_{\mathbb{R}^n} h\,d\gamma_n\right) \le \int_{\mathbb{R}^n} \sqrt{\mathcal{U}^2(h) + |\nabla h|^2}\,d\gamma_n. \tag{9.34}$$

It is not very difficult to see that (9.34) is equivalent to the isoperimetric statement. For instance, if in (9.34) we replace h by (an approximation of) the characteristic function of a set B, we find that

$$\mathcal{U}(\gamma(B)) \le S_\gamma(B),$$

which is precisely the Gaussian isoperimetry. Beckner's observation is the following: if we replace replace h by εh, expand it for ε close to 0, and use the fact that, for x close to 0,

$$\mathcal{U}(x) \sim x\sqrt{2\log(1/x)},$$

then (9.34) turns into the Stam-Gross logarithmic Sobolev inequality in the limit as $\varepsilon \to 0$.

In a further contribution, Bobkov and Ledoux [**50**] strengthened even more the links between isoperimetry and logarithmic Sobolev inequalities, by devising a proof of the Bakry-Émery theorem which only relies on the Brunn-Minkowski inequality (6.1).

• *Functional analysis on Riemannian manifolds.* The above remarks show that there is some geometric content in the logarithmic Sobolev inequality. In fact, these inequalities are sometimes very useful to study properties of Riemannian manifolds (spectral gaps for Laplace-Beltrami operators, etc.). There is a version of the Bakry-Émery theorem on a Riemannian manifold: inequality (9.27) holds true as soon as $\mathrm{Hess}\, V + \mathrm{Ric} \geq \lambda I_n$, where Ric is the Ricci tensor on the manifold, and n its dimension.

This line of ideas led Bakry and Ledoux to propose notions of curvature and dimension for a general diffusion operator [**179, 26, 182**], in terms of the optimal constants appearing in some Sobolev-type inequalities.

• *Information theory.* The relative informations of Kullback and Fisher are basic objects in information theory. A detailed account on related inequalities in this field, and of their links with logarithmic Sobolev inequalities, can be found in Dembo, Cover and Thomas [**115**] and in Carlen [**83**]. Also a very nice source is [**17**, Chapter 10]. Here we shall just mention two famous inequalities. In order to state them in an appealing form, we shall abuse notation by writing $H(X)$, $I(X)$ for $H(f)$, $I(f)$ when X is a random vector with law $f\,dx$.

- The **Blachman-Stam inequality** [**44, 241, 83**] states that for all $\alpha \in [0,1]$ and for all independent random vectors X, $\widetilde{X}$ in $\mathbb{R}^n$,

$$I\Big(\sqrt{\alpha}\, X + \sqrt{1-\alpha}\, \widetilde{X}\Big) \leq \alpha I(X) + (1-\alpha) I(\widetilde{X}).$$

Equivalently, whenever ρ and $\widetilde{\rho}$ are two probability measures,

$$\frac{1}{I(\rho * \widetilde{\rho})} \geq \frac{1}{I(\rho)} + \frac{1}{I(\widetilde{\rho})}.$$

- The **Shannon-Stam inequality** [**241, 84**] states that for all $\alpha \in [0,1]$ and for all independent random vectors X, $\widetilde{X}$ in $\mathbb{R}^n$,

$$H\Big(\sqrt{\alpha}\, X + \sqrt{1-\alpha}\, \widetilde{X}\Big) \leq \alpha H(X) + (1-\alpha) H(\widetilde{X}).$$

Equivalently, whenever ρ and $\widetilde{\rho}$ are two probability measures on $\mathbb{R}^n$,

$$\mathcal{N}(\rho * \widetilde{\rho}) \geq \mathcal{N}(\rho) + \mathcal{N}(\widetilde{\rho}).$$

The Blachman-Stam inequality can be considered as a reinforcement of the Stam-Gross logarithmic Sobolev inequality. In fact, a particular case of it (with one of the two densities being Gaussian) reads

$$I(\rho_t|\gamma) \leq e^{-2t} I(\rho_0|\gamma), \tag{9.35}$$

where ρ_t stands for the solution of the Fokker-Planck equation

$$\frac{\partial \rho}{\partial t} = \Delta \rho + \nabla \cdot (\rho x) \qquad (t \geq 0, x \in \mathbb{R}^n),$$

with initial datum ρ_0. Now, if one integrates inequality (9.35) in time, from 0 to $+\infty$, one obtains exactly (9.30), with ρ replaced by ρ_0. On the other hand, the Blachman-Stam inequality also implies the Shannon-Stam inequality by a similar semigroup argument. And, as already mentioned in Chapter 6, the Shannon-Stam inequality can also be considered as a limit case of Young's inequality for convolution (with optimal constants) — and, guess what, the logarithmic Sobolev inequality can also be derived from the optimal Young inequality, as shown in [**190**, Theorem 8.14]! At this point, the reader may have the feeling that all the inequalities which we have encountered are connected, which may be not so false. As explained in [**83, 34**], the Stam-Gross logarithmic Sobolev inequality can even be seen as a direct consequence of the so-called Beckner-Hirschman uncertainty principle [**166, 32**], which is a deep and beautiful generalization of the Heisenberg inequality in quantum mechanics. See also [**33**] for links with the Hardy-Littlewood-Sobolev inequality.

• *Hypercontractivity estimates.* Let $(P_t)_{t \geq 0}$ denote a semigroup, defined by an equation of the form

$$\frac{\partial}{\partial t}(P_t f) = \mathcal{L}(P_t f),$$

where $\mathcal{L}$ is self-adjoint in $L^2(d\rho_\infty)$. This semigroup is said to be **hypercontractive** if $P_t : L^p(d\rho_\infty) \longrightarrow L^{q(t)}(d\rho_\infty)$, with $q(t) > p$ when $t > 0$ and $p > 1$. Gross [**160**] showed that LSI(λ) is equivalent to the following hypercontractivity estimate on the semigroup (P_t) generated by the linear operator $\mathcal{L} = -L = \Delta - \nabla V \cdot \nabla$:

$$\|P_t h\|_{L^{q(t)}(d\rho_\infty)} \leq \|h\|_{L^p(d\rho_\infty)}, \qquad \frac{q(t)-1}{p-1} \leq e^{2\lambda t}. \tag{9.36}$$

The theory of hypercontractive semigroups was pioneered by Bonami [**53**], and later spectacularly pushed forward by Nelson (who proved hypercontractivity in the case of a quadratic potential V) and Gross. It has been the object of a lot of attention since the eighties. There are numerous extensions, including versions with negative Lebesgue exponents!

Bobkov, Gentil and Ledoux [**46**] showed that LSI(λ) is also equivalent to another type of hypercontractivity. Namely,

$$\|e^{Q_t u}\|_{L^{q(t)}(d\rho_\infty)} \leq \|e^u\|_{L^p(d\rho_\infty)}, \qquad q(t) = p + \lambda t, \tag{9.37}$$

where Q_t is the semigroup associated to the Hamilton-Jacobi equation

$$\frac{\partial}{\partial t}(Q_t u) + \frac{|\nabla(Q_t u)|^2}{2} = 0.$$

Equivalently, as we saw in Chapter 5, $Q_t u$ is given by the Hopf-Lax formula

$$Q_t u(x) = \inf_{y\in\mathbb{R}^n} \left(\frac{|x-y|^2}{2} + u(y) \right).$$

This work by Bobkov, Gentil and Ledoux was directly inspired by optimal transportation problems described in the next section, and may be consulted as an extremely clear reference for links between logarithmic Sobolev inequalities and hypercontractivity estimates.

• *Theory of concentration of measure.* The general question in the theory of concentration of measure is the following. Consider a reference probability measure ρ_∞ on some measure space X equipped with a distance d. Then consider a measurable set $B \subset X$ with positive measure, and define its thickening

$$B_t = \{x \in X; \quad d(x, B) \le t\}, \qquad d(x, B) = \inf_{y\in B} d(x, y). \tag{9.38}$$

The problem is to estimate, under various assumptions, how fast $\rho_\infty[B_t]$ converges to 1 as t becomes large.

This problem was first considered by Lévy, who proved that on the n-dimensional sphere with uniform measure, the measure of B_t^c decreases essentially like e^{-nt^2}. Lévy deduced that a Lipschitz function on a sphere of large dimension is "almost constant" except for a vanishingly small set. To have an idea of what this statement may precisely mean, the reader may have a look at Problem 15 in Chapter 10.

Later the point of view of concentration of measure was put forward by Gromov and Milman, with applications to geometrical problems or to the theory of Banach spaces. Concentration estimates have also turned out to be quite useful in several areas of probability, statistics and statistical physics. During the nineties, they have been spectacularly developed by Talagrand (see [**245, 246**]) in the case where ρ_∞ is a product measure. Also Talagrand has considered the problem in a more abstract framework, where the notion of "distance" is not always a metric one (see [**246**]). A summary of Talagrand's work can be found in [**180**].

Starting from a remark by Herbst, Ledoux and collaborators studied in very precise detail the connections between logarithmic Sobolev inequality and concentration of measure. A nice account on this line of research is the set of lecture notes [**181**]. Also by the same author, the book [**183**] is warmly recommended to the reader who wants to know more about this

topic, or about concentration of measure in general. An application of the links between concentration of measure and logarithmic Sobolev inequalities is presented in Problem 15 in Chapter 10, following the PhD thesis of Malrieu [**199**].

• *Entropy production estimates and trend to equilibrium for dissipative equations.* This is the problem by which we motivated the introduction of logarithmic Sobolev inequalities. It is easily understood why logarithmic Sobolev inequalities are useful here: if $\rho_\infty = e^{-V}$ satisfies a logarithmic Sobolev inequality, then any solution $(\rho_t)_{t\geq 0}$ of the Fokker-Planck equation

$$\frac{\partial \rho}{\partial t} = \Delta\rho + \nabla \cdot (\rho \nabla V)$$

satisfies

$$-\frac{d}{dt} H(\rho_t|\rho_\infty) \geq -2\lambda H(\rho_t|\rho_\infty),$$

hence

$$H(\rho_t|\rho_\infty) \leq H(\rho_0|\rho_\infty) e^{-2\lambda t},$$

which is an excellent estimate of trend to equilibrium (in fact optimal), using rather weak estimates on the initial datum.

The field of entropy production estimates has undergone spectacular development since the mid-nineties. Some of the most important trends were applications of logarithmic Sobolev inequalities to nonlinear partial differential equations (among many references, see the examples in [**20**]), the derivation of entropy-entropy production inequalities for equations of porous medium type [**218, 88, 112**], Fokker-Planck equations with degenerately convex confinement potential [**253**], or kinetic models [**117, 118, 252**]. A particularly tricky and interesting (well, in the taste of the author) topic is the study of entropy-entropy production inequalities for the Boltzmann equation, see [**252, 263, 264**].

In Section 9.6 below, we shall illustrate these issues by some results about simple kinetic equations for granular media.

Exercise 9.10 (Coupling solutions of diffusion equations). This exercise presents an alternative, more elementary treatment of convergence to equilibrium for the linear Fokker-Planck equation, having several similarities with our discussion about Tanaka's theorem in Chapter 7. It requires a little familiarity with the elementary theory of diffusion processes. Let V be a λ-uniformly convex, C^2 potential. It is well-known that if the stochastic process $(X_t)_{t\geq 0}$ solves the stochastic differential equation

$$dX_t = \sqrt{2}\, dB_t - \nabla V(X_t)\, dt,$$

where $(B_t)_{t\geq 0}$ is a Brownian process, then the law μ_t of X_t solves the linear Fokker-Planck equation:

$$\frac{\partial \mu_t}{\partial t} = \Delta \mu_t + \nabla \cdot (\nabla V \, \mu_t).$$

(i) Let μ_0 and ν_0 be two probability measures on $\mathbb{R}^n$ with finite second order moments, let $(\mu_t)_{t\geq 0}$ and $(\nu_t)_{t\geq 0}$ be the solutions of the Fokker-Planck equation starting from the initial data μ_0 and ν_0 respectively, and let $\pi_0 \in \Pi(\mu_0, \nu_0)$. Show that one can define stochastic processes X_t and Y_t, in such a way that $\text{law}(X_t) = \mu_t$, $\text{law}(Y_t) = \nu_t$, $\text{law}(X_0, Y_0) = \pi_0$, and

$$\forall t > 0, \qquad \frac{d}{dt}(X_t - Y_t) = -[\nabla V(X_t) - \nabla V(Y_t)].$$

(ii) Use this to show that

$$W_2(\mu_t, \nu_t) \leq W_2(\mu_0, \nu_0)\, e^{-\lambda t}. \tag{9.39}$$

(iii) Conclude that μ_t converges exponentially fast to the equilibrium.

Remark 9.11. This scheme of proof seems to fail as soon as we leave the setting of linear equations.

Exercise 9.12 (Explicit calculations in one dimension). Assuming $n = 1$, use the explicit formula for W_2, equation (2.47), to prove that

$$\left.\frac{d}{dt}\right|^{+} W_2(\mu_t, \nu_t)^2 \leq -2\lambda\, W_2(\mu_t, \nu_t)^2,$$

and recover (9.39). This is not so easy as it seems!

9.2.2. The Bakry-Émery argument. The original proof of the Bakry-Émery theorem was schematically as follows. Let $(\rho_t)_{t\geq 0}$ be a solution of the Fokker-Planck equation

$$\frac{\partial \rho}{\partial t} = \Delta \rho + \nabla \cdot (\rho \nabla V),$$

where $\int e^{-V} = 1$ and $D^2 V \geq \lambda I_n$. Then one computes the dissipation of $H(\rho_t | e^{-V})$, and the dissipation of the dissipation itself:

$$D(\rho_t) = \int \rho_t |\xi_t|^2,$$

$$DD(\rho_t) = 2 \int \rho_t \, \text{tr}\,[(\nabla \xi)^T \nabla \xi] + 2 \int \rho_t \, \langle D^2 V \cdot \xi_t, \xi_t \rangle, \tag{9.40}$$

where

$$\xi_t = -\nabla(\log \rho_t + V).$$

It follows that $DD(\rho_t) \geq 2\lambda D(\rho_t)$, and when one integrates this with respect to t one recovers $D(\rho_0) \geq 2\lambda H(\rho_0 | e^{-V})$.

The proof of Bakry and Émery actually does not use this formalism. The preceding formulas are closer to the way the Bakry-Émery proof was rewritten by Arnold, Markowich, Toscani and Unterreiter [**20**] at the end of the nineties. All the hard work in this argument consists in establishing (9.40).

9.2.3. Optimal transportation and logarithmic Sobolev inequalities. Up to a few years ago, the only available proof of the Bakry-Émery theorem was based on the above argument, involving the Fokker-Planck semigroup. Then, in the two years 1999 and 2000, four different, new arguments were discovered, three of which were based on optimal transportation! These proofs can be found in a paper by Otto and the author [**219**] (later simplified by Cordero-Erausquin [**97**]), in Bobkov, Gentil and Ledoux [**46**], and finally in Caffarelli [**79**]. The fourth argument which we have in mind is the proof by Bobkov and Ledoux [**50**], entirely based on the Brunn-Minkowski inequality — and even this proof is not really transport-free, since we have seen in Chapter 6 how the Brunn-Minkowski inequality can be proven in a straightforward way by optimal transportation methods.

This appearance of optimal transportation may seem surprising, since it is completely absent from the definition of logarithmic Sobolev inequalities and from the Bakry-Émery argument. However, it becomes much more natural if one recalls Theorem 9.2. In fact, the combination of Theorem 5.15 (ii) and of Theorem 9.2 (i) formally implies the Bakry-Émery theorem at once. Thus the latter can be seen in a very intuitive way as a manifestation of the uniform displacement convexity of the relative information functional. In Section 9.4 below, we shall make this remark more precise and provide a complete proof.

Before closing the present subsection, we wish to say a few words about Caffarelli's approach [**79**]. He establishes the following deep and surprisingly general theorem:

Theorem 9.13 (Gaussian Sobolev-type inequalities improve by log-concave perturbation). *Let F, G, H, J, L be nonnegative continuous functions on $\mathbb{R}$, with H and J nondecreasing. Then a functional inequality of the form* (9.41)

$$\int_{\mathbb{R}^n} L(h)\, d\rho_\infty = \ell \quad \Longrightarrow \quad F\left(\int_{\mathbb{R}^n} G(|h|)\, d\rho_\infty\right) \leq \frac{1}{\lambda} H\left(\int_{\mathbb{R}^n} J(|\nabla h|)\, d\rho_\infty\right),$$

where ρ_∞ is a reference probability measure, $\ell \in \mathbb{R}$ and h is arbitrary, can only be improved by a log concave perturbation of the Gaussian.

More explicitly, let $\lambda(\rho_\infty)$ be the largest admissible constant λ in (9.41). If γ is the standard Gaussian measure and $\rho_\infty = e^{-v}\gamma$ with v convex, then $\lambda(\rho_\infty) \geq \lambda(\gamma)$.

To recover the Bakry-Émery theorem from Theorem 9.13, set $F(s) = s$, $G(s) = s^2 \log s^2 - s^2 + 1$, $H(s) = 2s$, $J(s) = s^2$, $L(s) = s^2$, $\ell = 1$; then one can recognize inequality (9.32). This implies the Bakry-Émery theorem only when $\lambda = 1$, but then the general case follows by a rescaling argument.

The proof of Theorem 9.13 is an immediate consequence of the following remarkable fact about optimal transportation:

Theorem 9.14 (Caffarelli's log concave perturbation theorem). *Let v be a C^2 convex potential on $\mathbb{R}^n$, let γ be the standard Gaussian measure and let $\rho_\infty = e^{-v}\gamma$. Let $T = \nabla\varphi$ be the unique gradient of a convex map such that $\nabla\varphi\#\gamma = \rho_\infty$. Then,*

$$0 \leq D^2\varphi \leq I_n.$$

Theorem 9.14 itself is a consequence of the fact that second derivatives of solutions of the Monge-Ampère equation are subsolutions of an elliptic equation. An idea of the proof is given in Problem 13 in Chapter 10.

We now explain how Theorem 9.13 follows from Theorem 9.14. First, observe that

$$\int G(h)\, d\rho_\infty = \int G(h \circ T)\, d\gamma, \qquad \int L(h)\, d\rho_\infty = \int L(h \circ T)\, d\gamma$$

(by definition of push-forward). Next, we can write

$$\int J(|\nabla h|)\, d\rho_\infty = \int J(|\nabla h \circ T|)\, d\gamma \geq \int J(|\nabla(h \circ T)|)\, d\gamma,$$

where the last inequality comes from Theorem 9.14, in the form

$$|\nabla(h \circ T)| = \left|\langle D^2\varphi \cdot (\nabla h \circ \nabla\varphi)\rangle\right| \leq |\nabla h \circ \nabla\varphi|,$$

and the nondecreasing property of J. From this the conclusion is immediate.

9.3. Talagrand inequalities

Let ρ_∞ be a reference probability measure. We say that it satisfies the inequality $\mathbf{T}(\lambda)$ ("T" may be understood either as "Talagrand", or as "transportation") if

$$\forall \rho \in P(\mathbb{R}^n), \qquad W_2(\rho, \rho_\infty) \leq \sqrt{\frac{2}{\lambda} H(\rho|\rho_\infty)}. \tag{9.42}$$

9.3.1. Background.

Inequalities such as (9.42) were introduced by Talagrand [**247**] in the case when ρ_∞ is the Gaussian γ. Talagrand took advantage of the tensorization properties of both the Gaussian and the quadratic Wasserstein distance to prove the functional inequality (9.42) by induction on the dimension n.

A refinement of Talagrand's method led Blower [**45**] to prove that

$$D^2 V \geq \lambda I_n \Longrightarrow T(\lambda).$$

There is however a stronger statement in the slightly earlier work [**219**]:

$$LSI(\lambda) \Longrightarrow T(\lambda).$$

This kind of implication between logarithmic Sobolev inequalities and transportation inequalities was first conjectured by Bobkov and Götze [**48**]. The interest of this refinement lies in the relative flexibility allowed by logarithmic Sobolev inequalities, as compared with Talagrand inequalities. Indeed, the Wasserstein distance is sometimes quite tricky to handle, because of its nonlocal character. For instance, at the moment, no analog of the Holley-Stroock perturbation result (Theorem 9.9 (ii)) has been proven for Talagrand inequalities.

It is not known whether the converse implication is true. More precisely: assuming that T(λ) holds true, then does ρ_∞ automatically satisfy a logarithmic Sobolev inequality, maybe with worse constants? As shown in [**219**], the answer is yes if V is convex; we shall explain this in the next section. Surprisingly, the combination of LSI(λ) and T(λ), i.e. the functional inequality

$$W_2(\rho, \rho_\infty) \leq \frac{\sqrt{I(\rho|\rho_\infty)}}{\lambda},$$

is equivalent to the logarithmic Sobolev inequality, up to changing the constants by a multiplicative factor which only depends on a lower bound for the Hessian matrix $D^2 V$.

9.3.2. Concentration estimates. Let us briefly evoke the main motivation of Talagrand's study, namely the theory of concentration of measure.

Building on an argument by Marton, Talagrand noticed that such inequalities as (9.42) lead to elementary (and often almost sharp) concentration inequalities. For instance, let $B \subset \mathbb{R}^n$, $\rho_\infty[B] > 0$, and consider its thickening B_t defined as in (9.38). Moreover, let $\rho_\infty|_A$ denote the restriction of ρ_∞ to the measurable set A, i.e. the probability measure

$$\rho_\infty|_A = \frac{1_A \, \rho_\infty}{\rho_\infty[A]}.$$

Of course,

$$W_2\Big(\rho_\infty|_B, \rho_\infty|_{B_t^c}\Big) \geq t. \tag{9.43}$$

Indeed, in the transportation process from one probability measure to the other, all of the measure has to go from B to B_t^c, which means that each

unit of mass travels a distance at least t. On the other hand, by the triangle inequality,

$$W_2\Big(\rho_\infty|_B, \rho_\infty|_{B_t^c}\Big) \le W_2\Big(\rho_\infty|_B, \rho_\infty\Big) + W_2\Big(\rho_\infty|_{B_t^c}, \rho_\infty\Big). \tag{9.44}$$

By combining (9.43), (9.44) and (9.42), and noting that $H(\rho_\infty|_B \,|\, \rho_\infty) = \log(1/\rho_\infty[B])$, one gets

$$t \le \sqrt{\frac{2}{\lambda}\log\frac{1}{\rho_\infty[B]}} + \sqrt{\frac{2}{\lambda}\log\frac{1}{\rho_\infty[B_t^c]}},$$

or equivalently

$$\rho_\infty[B_t] \ge 1 - e^{-\frac{\lambda}{2}\left(t - \sqrt{2\log\frac{1}{\rho_\infty[B]}}\right)^2}.$$

For example, if we recall that the Gaussian satisfies a Talagrand inequality with constant 1, we see that

$$\gamma[B_t] \ge 1 - e^{-\frac{1}{2}\left(t - \sqrt{\frac{2}{\lambda}\log\frac{1}{\gamma[B]}}\right)^2}.$$

This is a typical result in the theory of concentration of measure. It is not optimal (optimal results can be obtained via the study of the Gaussian isoperimetry), but the factor $1/2$ is sharp and the inequality is quite close to be optimal.

Further links between transportation inequalities in the same style and concentration inequalities were investigated in great detail by Bobkov and Götze [**48**]. One of their main results is that a concentration inequality similar to the one which is enjoyed by the Gaussian measure (see equation (10.34) below) is *equivalent* to the functional inequality

$$W_1(\rho, \rho_\infty) \le \sqrt{\frac{2H(\rho|\rho_\infty)}{\lambda}}. \tag{9.45}$$

Of course, this inequality itself is a priori weaker than the Talagrand inequality $\mathrm{T}(\lambda)$, since $W_1 \le W_2$. See [**120**] for further results on this subject.

9.3.3. Relative entropy control.

When it applies, inequality (9.42) implies that results of convergence in relative entropy are stronger than results of convergence in the quadratic Wasserstein sense. A remark is in order here. The Csiszár-Kullback-Pinsker inequality, rewritten as

$$H(\rho|\rho_\infty) \ge \frac{1}{2}\|\rho - \rho_\infty\|_{TV}^2, \tag{9.46}$$

seems to provide a stronger notion of control. This however is not true in all circumstances, in particular because inequality (9.46) results in a higher and higher loss of information when the dimension n becomes larger and larger. To understand this, just note that the left-hand side in (9.46) is "typically"

of the order of the dimension (this is the case for instance if ρ_∞ is an n-fold tensor product), while the right-hand side is always bounded by 2!

9.3.4. Proofs. There are, at present, two different available arguments for the implication LSI(λ) $\Rightarrow$ T(λ). The original one, due to Otto and the author [**219**], is the implementation of the abstract proof shown in subsection 9.1.4. It is somewhat more complicated and slightly less general than the other one, but may be appealing to the reader with a mind oriented to partial differential equations, so we shall sketch it.

The starting point is again the Fokker-Planck equation

$$\frac{\partial \rho}{\partial t} = \Delta \rho + \nabla \cdot (\rho \nabla V).$$

From parabolic regularity estimates and the strong maximum principle, one shows (under the assumptions $V \in C^2(\mathbb{R}^n)$, $D^2 V \geq -CI_n$, $C \in \mathbb{R}$) that $\log \rho$ is a smooth function, so this equation can be rewritten as a nonlinear transport equation,

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \xi) = 0, \qquad \xi(t, \cdot) = -\nabla \left(\log \frac{\rho}{\rho_\infty} \right).$$

Then one can apply Theorem 5.34 in order to solve this equation a posteriori by the method of characteristics:

$$\rho_t = T_t \# \rho_0, \qquad \frac{\partial T_t}{\partial t} = \xi_t \circ T_t, \qquad T_0 = \mathrm{Id}\,. \tag{9.47}$$

Here we say "a posteriori", because the characteristic equations depend on ρ_t itself, so this method cannot be used to predict the evolution of ρ_t.

As a consequence of (9.47) and the definition of optimal transportation,

$$W_2(\rho_0, \rho_t) \leq \sqrt{\int_{\mathbb{R}^n} |T_t(x) - x|^2 \, d\rho_0(x)}.$$

Thus,

$$\left.\frac{d}{dt}\right|_{t=0}^{+} W_2(\rho_0, \rho_t) \leq \limsup_{t \to 0} \sqrt{\int_{\mathbb{R}^n} \left| \frac{T_t(x) - x}{t} \right|^2 d\rho_0(x)}. \tag{9.48}$$

But $T_t x - x = T_t x - T_0 x = t\, \xi_0(x) + o(t)$. By dominated convergence, one can deduce that the right-hand side of (9.48) converges to

$$\sqrt{\int_{\mathbb{R}^n} |\xi_0|^2 \, d\rho_0} = \sqrt{I(\rho_0 | \rho_\infty)}.$$

More generally,

$$\left.\frac{d}{ds}\right|_{s=0}^{+} W_2(\rho_t, \rho_{t+s}) \leq \sqrt{I(\rho_t | \rho_\infty)}.$$

Then one has all the ingredients to conclude the proof as in subsection 9.1.4; in particular, the convergence $H(\rho_t|\rho_\infty) \longrightarrow 0$ as $t \to \infty$ is ensured by the fact that ρ_∞ satisfies a logarithmic Sobolev inequality.

The second argument, due to Ledoux and coworkers, is quite different. It relies on a **dual formulation** of the Talagrand inequality. We give it as an exercise.

Exercise 9.15 (A proof of LSI ⇒ T via duality). Recall from Chapter 1 that

$$W_2(\rho,\rho_\infty)^2 = \sup\left\{\int \varphi\, d\rho + \int \psi\, d\rho_\infty;\quad \varphi(x)+\psi(y) \le |x-y|^2\right\}. \tag{9.49}$$

(i) Establish the Legendre-type representation formula

$$\begin{aligned} H(\rho|\rho_\infty) &= \sup_{f\in C_0(\mathbb{R}^n)} \left\{\int f\, d\rho;\quad \int e^f \rho_\infty \le 1\right\} \\ &= \sup_{f\in C_0(\mathbb{R}^n)} \left\{\int f\, d\rho - \log\left(\int e^f\, d\rho_\infty\right)\right\}. \end{aligned} \tag{9.50}$$

(ii) Based on (9.49) and (9.50), guess the following **dual formulation for $\mathbf{T}(\lambda)$:**

(9.51)

$$\forall g \in C_b(\mathbb{R}^n), \qquad \int e^{\lambda \inf_y \left[g(y)+\frac{|x-y|^2}{2}\right]}\, d\rho_\infty(x) \le \exp\left(\lambda \int g\, d\rho_\infty\right).$$

This dual formulation is detailed in Rachev [**226**].

(iii) Let $(Q_t)_{t\ge 0}$ denote the Hamilton-Jacobi semigroup, induced by the equation $\partial u/\partial t + |\nabla u|^2/2 = 0$. Rewrite equation (9.51) as

$$\frac{1}{\lambda}\log\left(\int e^{\lambda Q_1 g}\, d\rho_\infty\right) \le \int g\, d\rho_\infty.$$

Hint: Recall subsection 5.4.6 in Chapter 5.

(iv) Assuming that g satisfies adequate integrability assumptions, check that

$$\lim_{t\downarrow 0} \frac{1}{\lambda t}\log\left(\int e^{\lambda t Q_t g}\, d\rho_\infty\right) = \int g\, d\rho_\infty.$$

(v) Hence, to prove (9.51), it is sufficient to prove that the function

$$\phi(t) = \frac{1}{t}\log\left(\int e^{\lambda t Q_t g}\, d\rho_\infty\right)$$

is nonincreasing as t goes from 0 to 1. Show that this monotonicity property can be rewritten, at least formally, as

$$\lambda t \int e^{\lambda t Q_t g} Q_t g \, d\rho_\infty - \left(\int e^{\lambda t Q_t g} \, d\rho_\infty\right) \log\left(\int e^{\lambda t Q_t g} \, d\rho_\infty\right) \leq \frac{\lambda t^2}{2} \int e^{\lambda t Q_t g} |\nabla Q_t g|^2 \, d\rho_\infty,$$

and show that this functional inequality is true if ρ_∞ satisfies LSI(λ).

Hint: Use the Hamilton-Jacobi equation explicitly.

Remark 9.16. A little bit of justification is needed at the end of the above exercise, because solutions of the Hamilton-Jacobi equation are not smooth. But the fact that the equation is satisfied almost everywhere is sufficient to conclude, see [**46**]. We do not wish to insist on this point here.

Just as the logarithmic Sobolev inequality, the Talagrand inequality T(λ) implies the Poincaré inequality P(λ) by a linearization argument [**219**]. This is natural if one recalls that the "linearization" of the quadratic Wasserstein distance is the H^{-1} norm, as we discussed in Section 7.6. Interestingly enough, the implication T(λ) $\Rightarrow$ P(λ) is obtained in [**46**] by using the approximation $Q_t u = u - t|\nabla u|^2/2 + o(t)$.

Further investigations about the Hamilton-Jacobi semigroup method led the authors of [**46**] to discover the hypercontractivity formulation of logarithmic Sobolev inequalities which is mentioned in formula (9.37). The link between the hypercontractivity of the Fokker-Planck semigroup and that of the Hamilton-Jacobi semigroup is made clear in [**46**]: it goes through the idea of "vanishing viscosity". The key idea is that if h satisfies the equation

$$\frac{\partial h}{\partial t} = \mathcal{L}h \equiv \Delta h - \nabla V \cdot \nabla h,$$

then for any $\varepsilon > 0$, $u_\varepsilon(t,x) = -2\varepsilon \log h(\varepsilon t, x)$ is a solution of the **viscous Hamilton-Jacobi equation**

$$\frac{\partial u_\varepsilon}{\partial t} + \frac{|\nabla u_\varepsilon|^2}{2} = \varepsilon \mathcal{L} u_\varepsilon.$$

They also found out that this method gave them a lot of flexibility, and with its help they recovered and generalized many inequalities appearing in the study of non-uniformly convex potentials, like modified logarithmic Sobolev inequalities, or some concentration inequalities due to Talagrand [**247**]. As an example, consider the "exponential" measure $\rho_\infty(x) = Z^{-1}\exp(-\sum_{i=1}^n |x_i|)$ on $\mathbb{R}^n$, where Z is a normalizing constant. For this

measure, an inequality like (9.42) does not hold; on the other hand there is a substitute which reads

$$\mathcal{T}_c(\rho, \rho_\infty) \le CH(\rho|\rho_\infty),$$

in which $\mathcal{T}_c$ is the transportation cost associated to the cost function

$$c(|x-y|) = \sum c(|x_i - y_i|),$$

where $c(|z|) = |z|^2$ for $|z| \le 1$, $c(|z|) = 2|z|$ for $|z| \ge 1$.

Further work about connections between these inequalities and Hamilton-Jacobi equations can be found in the note [**153**].

9.4. HWI inequalities

We finally turn to the study of HWI inequalities. We first state one of the main results in [**219**]. It is formally analogous to the abstract statement in Theorem 9.2 (iii).

Theorem 9.17 (HWI inequalities). *Let $\rho_\infty = e^{-V}$ be a reference probability distribution on $\mathbb{R}^n$, such that $D^2V \ge KI_n$ for some $K \in \mathbb{R}$, and $\int Ve^{-V} < +\infty$. Then*
(9.52)

$$\forall \rho \in P_{\mathrm{ac},2}(\mathbb{R}^n), \qquad H(\rho|\rho_\infty) \le W_2(\rho|\rho_\infty)\sqrt{I(\rho|\rho_\infty)} - \frac{K}{2}W_2(\rho,\rho_\infty)^2.$$

A variant of this inequality first appeared in Otto [**218**] for the study of porous medium diffusion equations. Let us mention the following more general variant of (9.52):

(9.53) $\forall \rho_0, \rho_1 \in P_{\mathrm{ac},2}(\mathbb{R}^n)$,

$$H(\rho_0|\rho_\infty) \le H(\rho_1|\rho_\infty) + W_2(\rho_0,\rho_1)\sqrt{I(\rho_0|\rho_\infty)} - \frac{K}{2}W_2(\rho_0,\rho_1)^2.$$

So equation (9.52) is the specialization of (9.53) when $\rho_1 = \rho_\infty$. Note that the choice $\rho_0 = \rho_\infty$ in (9.53) also leads to an interesting inequality (which one?). In the sequel we shall actually prove (9.53). But first we make a few comments on the significance of this inequality.

In order to get a better intuition for (9.52), consider the case $K = 0$ and linearize the inequality by the usual procedure $\rho = (1+\varepsilon f)\rho_\infty$, as presented in Section 7.6. Check that one recovers the inequality

$$\int f\rho_\infty = 0 \quad \Longrightarrow \quad \|f\|^2_{L^2(d\rho_\infty)} \le 2\|f\|_{H^{-1}(d\rho_\infty)}\|f\|_{\dot{H}^1(d\rho_\infty)}.$$

This inequality is suboptimal in the sense that the constant 2 can be improved into a constant 1. However, from this argument we see that the HWI inequality acts somewhat like an **interpolation inequality**. It gives us a

control in the sense of Kullback information, with the help of a control in the quadratic Wasserstein sense and a control in the Fisher information sense. As a typical example, let $(\rho_k)_{k\in\mathbb{N}}$ be a sequence of probability densities satisfying

$$W_2(\rho_k,\rho_\infty)\xrightarrow[k\to\infty]{}0,\qquad \sup_{k\in\mathbb{N}} I(\rho_k|\rho_\infty)<+\infty;$$

then, if the HWI inequality applies, it will follow that

$$H(\rho_k|\rho_\infty)\xrightarrow[k\to\infty]{}0.$$

This is a quantitative version of the general principle that weak convergence, combined with a smoothness bound, implies strong convergence. This inequality has proven useful for establishing various results of strong compactness, or convergence to equilibrium in the strong sense [**85**].

Let us see some direct consequences of (9.52). First, if $V=-\log\rho_\infty$ is convex, then it holds true with $K=0$. Then, if e^{-V} satisfies a Talagrand inequality $\mathrm{T}(\lambda)$, we can combine this information with (9.52) and recover in the end

$$H(\rho|\rho_\infty)\le\frac{2}{\lambda}I(\rho|\rho_\infty).$$

What we have just shown is that

$$\left.\begin{array}{c} e^{-V}\text{ satisfies }\mathrm{T}(\lambda)\\ D^2V\ge 0\end{array}\right\}\Longrightarrow e^{-V}\text{ satisfies }\mathrm{LSI}(\lambda/4).$$

Our second remark is that the Bakry-Émery theorem (point (i) of Theorem 9.9) is an immediate corollary of inequality (9.52). Indeed, if $K>0$, then, by Young's inequality $AB\le A^2/2+B^2/2$, we find that

$$W_2(\rho,\rho_\infty)\sqrt{I(\rho|\rho_\infty)}-\frac{K}{2}W_2(\rho,\rho_\infty)^2\le\frac{1}{2K}I(\rho|\rho_\infty).$$

We conclude this section by a proof of (9.52).

Proof of Theorem 9.17. We shall prove precisely the following. Let V be a semi-convex function, bounded from below, satisfying $D^2V\ge KI_n$, and let $F(\rho)=\int\rho\log\rho+\int\rho V$. Let $\rho_0,\rho_1\in P_{\mathrm{ac},2}(\mathbb{R}^n)$ be such that ρ_1 is bounded from above, and bounded from below by a positive constant on each compact set, and $F(\rho_1)<+\infty$. Then, inequality (9.53) holds true; more precisely, if we set $h=d\rho_0/d\rho_1$, then

$$F(h\rho_1)\le F(\rho_1)+W_2(h\rho_1,\rho_1)\sqrt{4\int|\nabla\sqrt{h}|^2\,d\rho_1}-\frac{K}{2}W_2(h\rho_1,\rho_1)^2, \tag{9.54}$$

which is (9.53) in disguise. Of course, inequality (9.52) will be the special case $\rho_1=e^{-V}$.

Step 1: Regularization. As our first step, we shall show that it is sufficient to prove (9.53) when both ρ_0 and ρ_1 are smooth and compactly supported, positive in the interior of their respective supports. The general case will indeed follow from this particular case by a somewhat tedious density argument, which we shall sketch briefly now. The reader may skip this at first reading.

What we have to prove is inequality (9.54) under the assumptions $h \geq 0$, $\int h(x)|x|^2\, d\rho(x) < +\infty$, $\int h\, d\rho_1 = 1$. If $\nabla\sqrt{h}$ does not lie in $L^2(d\rho_1)$, there is nothing to prove. It it does, then it is possible to find a sequence of compactly supported functions $h_k = \lambda_k h\chi(\cdot/k)$ (here χ is a smooth, compactly supported cut-off function, $0 \leq \chi \leq 1$, $\chi \equiv 1$ on $B(0,1)$) such that $\int h_k\, d\rho_1 = 1$ (for this we just have to choose λ_k properly), $\int |\nabla\sqrt{h_k}|^2\, d\rho_1 \longrightarrow \int |\nabla\sqrt{h}|^2\, d\rho_1$ (as a consequence of the dominated convergence theorem), $F(h\rho_1) \leq \liminf F(h_k\rho_1)$ (as a consequence of Fatou's lemma), $\int h_k(x)|x|^2\, d\rho(x) \longrightarrow \int h(x)|x|^2\, d\rho(x)$ (by dominated convergence again), and $W_2(h_k\rho_1, \rho_1) \longrightarrow W_2(h\rho_1, \rho_1)$ (by weak convergence and convergence of second-order moments). So we just have to prove the theorem when h is compactly supported. In this case, we know that $\nabla\sqrt{h} \in L^2(\mathbb{R}^n)$, since ρ_1 is locally bounded below by a positive constant.

By regularizing h into $h_\delta = \lambda_\delta(\sqrt{h} * \zeta_\delta)^2$, where $(\zeta_\delta)_{\delta>0}$ is a family of regularizing kernels (i.e. $\zeta_\delta = \delta^{-n}\zeta(\cdot/\delta)$, $\zeta \geq 0$, $\zeta \in \mathcal{D}$, $\int \zeta = 1$), we find that $\int h_\delta\, d\rho_1 = 1$ (by a suitable choice of λ_δ), $\int |\nabla\sqrt{h_\delta}|^2\, d\rho_1$ converges to $\int |\nabla\sqrt{h}|^2\, d\rho_1$ (because ρ_1 is bounded and $|\nabla\sqrt{h_\delta}|^2 = |\nabla\sqrt{h} * \zeta_\delta|^2$ converges in L^1 to $|\nabla\sqrt{h}|^2$), and $F(h\rho_1) \leq \liminf F(h_\delta\rho_1)$ (again by Fatou's lemma). So we can assume that h is compactly supported and *smooth*.

Then we truncate ρ_1 into $\rho_{1,k} = \lambda_k\rho_1\chi(\cdot/k)$ as before, and we see that $\lambda_k \to 1$, $F(\rho_{1,k}) \longrightarrow F(\rho_1)$ (by dominated convergence), $\int |\nabla\sqrt{h}|^2\, d\rho_{1,k} \longrightarrow \int |\nabla\sqrt{h}|^2\, d\rho_1$ (by weak convergence), $F(h\rho_1) \leq \liminf F(h\rho_{1,k})$ (by Fatou again), $\int h|x|^2\, d\rho_{1,k}(x) \longrightarrow \int h|x|^2\, d\rho_1(x)$, $\int |x|^2\, d\rho_{1,k}(x) \longrightarrow \int |x|^2\, d\rho_1(x)$, and $W_2(h\rho_{1,k}, \rho_{1,k}) \longrightarrow W_2(h\rho_1, \rho_1)$ (by weak convergence). So we just have to prove the theorem when ρ_1 is compactly supported.

Finally we regularize ρ_1 into $\rho_{1,\delta}$ by setting $\rho_{1,\delta} = \zeta_\delta * \rho_1$, and we see that the same identities hold true (to prove that $F(\rho_{1,\delta}) \longrightarrow F(\rho_1)$, note that $H(\rho_1) \leq \liminf_{\delta\downarrow 0} H(\rho_{1,\delta}) \leq H(\rho_1)$, where the first inequality follows from Fatou's lemma and the second one from Jensen's inequality). In the end, we conclude that we just have to prove (9.53) when *ρ_1 and $\rho_0 = h\rho_1$ are smooth and compactly supported*, positive almost everywhere in their supports.

Step 2: Displacement interpolation. In this case, we define $\nabla\varphi$ to be the "unique" gradient of a convex function such that $\nabla\varphi\#\rho_0 = \rho_1$, and we introduce McCann's interpolant $(\rho_t)_{0\leq t\leq 1}$ between ρ_0 and ρ_1.

Let $F(\rho) = \int \rho \log \rho + \int \rho V$, and assume that V satisfies $D^2 V \geq K I_n$. From Theorem 5.15 we know that F inherits this property, in the sense that $(d^2/dt^2) F(\rho_t) \geq K W_2(\rho_0, \rho_1)^2$. Using Theorem 5.30, we deduce that

$$F(\rho_1) \geq F(\rho_0) - \int (\Delta_A \varphi - n)\, d\rho_0 + \int \langle \nabla V(x), \nabla\varphi(x) - x \rangle\, d\rho_0(x) + \frac{K}{2} W_2(\rho_0, \rho_1)^2.$$

To conclude the proof, it suffices to prove that

$$\int (\Delta_A \varphi - n)\, d\rho_0 - \int \langle \nabla V(x), \nabla\varphi(x) - x \rangle\, d\rho_0(x) \leq W_2(\rho_0, \rho_1) \sqrt{I(\rho_0|\rho_\infty)}.$$

For this we shall integrate by parts in the first integral. Since ρ_1 is compactly supported and $\nabla\varphi(\mathrm{Supp}(\rho_0)) \subset \mathrm{Supp}(\rho_1)$, we know that $\nabla\varphi$ is bounded on the support of ρ_0, so that φ is uniformly Lipschitz there. This makes it possible to extend φ into a convex Lipschitz function on the whole of $\mathbb{R}^n$ (exercise). Then the inequality $\Delta_A \varphi \leq \Delta_{\mathcal{D}'} \varphi$ is valid on the whole of $\mathbb{R}^n$, and we can write

$$\int (\Delta_A \varphi - n)\, d\rho_0 \leq \int (\Delta_{\mathcal{D}'} \varphi - n)\, d\rho_0 = -\int \langle \nabla\varphi(x) - x, \nabla \rho_0(x) \rangle\, dx.$$

Since ρ_0 is positive in the interior of its support, we can write $\nabla \rho_0 = (\nabla \log \rho_0)\rho_0$ almost everywhere, and therefore

$$\int (\Delta_A \varphi(x) - n)\, d\rho_0(x) \leq -\int \langle \nabla\varphi(x) - x, \nabla \log \rho_0(x) \rangle\, d\rho_0(x).$$

By the Cauchy-Schwarz inequality, we conclude that

$$\int (\Delta_A \varphi - n)\, d\rho_0 - \int \langle \nabla V(x), \nabla\varphi(x) - x \rangle\, d\rho_0(x) \leq \sqrt{\int |x - \nabla\varphi(x)|^2\, d\rho_0(x)} \sqrt{\int |\nabla \log \rho_0(x) + \nabla V(x)|^2\, d\rho_0(x)},$$

which was our goal. $\square$

This proof is quite close to the one given by Cordero-Erausquin [**97**]. A variant can be found in [**219**]. Again, the reader should not be abused by the technicalities, and rewrite for himself or herself the core of the proof in an informal manner, without worrying about rigorous justifications.

With this we conclude our overview of what should by now be considered as part of the "classical" theory of these functional inequalities. In the next few pages, we shall present some generalizations of interest, taken from more recent research papers, whose development has been associated with optimal transportation.

9.5. Nonlinear generalizations: internal energy

Let F be a λ-uniformly displacement convex functional ($\lambda > 0$). By Theorem 9.2, one expects the functional inequality

$$\|\operatorname{grad}_W F(\rho)\|_W^2 \geq 2\lambda\big[F(\rho) - \inf F\big] \tag{9.55}$$

to hold true.

As a particular case of Theorem 5.15, the functional

$$F(\rho) = \frac{1}{m-1}\int \rho(x)^m\,dx + \int \rho(x)\frac{|x|^2}{2}\,dx \tag{9.56}$$

is 1-uniformly displacement convex, as soon as $m \geq 1-1/n$ (when $m = 1$, the power function $U(s) = s^m/(m-1)$ should be replaced by $U(s) = s\log s$ and we would be back to the previous discussion about Kullback information). From Section 6.4 we know that this limit exponent $1 - 1/n$ is related to the critical exponent in the Sobolev embedding.

Thus, we expect the following inequalities to hold true for $m \geq 1 - 1/n$:

$$\int \left|\frac{m}{m-1}\nabla\rho^{m-1}(x) + x\right|^2 d\rho(x) \geq 2\,[F(\rho) - F(\rho_\infty)], \tag{9.57}$$

where ρ_∞ is the minimizer of F (or equivalently, inequality (9.57) should hold true for all densities ρ_∞). A little bit of work yields

$$\rho_\infty(x) = \left(C^2 - \frac{m-1}{2m}|x|^2\right)_+^{\frac{1}{m-1}}, \tag{9.58}$$

where C is a normalizing constant for ρ_∞ to be a probability measure. Thus, for $m > 1$ the minimizer has compact support; for $m < 1$ it decays polynomially at infinity; and for $m = 1$, the intermediate situation, it decays exponentially fast at infinity. This probability distribution is known as the **Barenblatt-Pattle profile**. It is important to note that the case $n = 2$, $m = 1/2$ should be excluded because ρ_∞ given by (9.58) has infinite second-order moments.

These inequalities can indeed be proven. This has been done independently by several authors: Carrillo and Toscani [**88**] for the case $m > 1$, Del Pino and Dolbeault [**112, 113**] and Otto [**218**]. The proofs in [**88**] and in [**218**] consist in an adaptation of the Bakry-Émery strategy; for this Otto was relying on the intuition provided by the Riemannian formalism in Chapter 8. The proof in [**113**] is different, and we shall come back to this in a moment. In any case, the reader should have little difficulty in finding a proof based on the methods already introduced in these notes.

Exercise 9.18. Adapt the proof of the HWI inequality given above into a simple proof of (9.57) (you may admit the approximation argument to reduce to the case of smooth densities).

The family of inequalities (9.57) is quite interesting. It implies an exponential rate of convergence to equilibrium for solutions of the porous medium equation with drift

$$\frac{\partial \rho}{\partial t} = \Delta \rho^m + \nabla \cdot (\rho x).$$

With the help of rescaling arguments and a bit of regularity theory for porous medium equations, this has enabled various authors to establish precise asymptotics for solutions of the porous medium equations

$$\frac{\partial \rho}{\partial t} = \Delta \rho^m \tag{9.59}$$

as $t \to \infty$. One finds that these solutions are asymptotic to self-similar solutions which are built from the Barenblatt-Pattle profiles (9.58) (see the above-mentioned references).

But the story does not end here. An amazing connection with a quite different family of functional inequalities was noticed by Del Pino and Dolbeault [**114, 113**]. This connection illustrates again the tight links between all these functional inequalities with geometric content; moreover it was one of the motivations for the study of the optimal Sobolev inequalities by means of transportation methods, described in Chapter 6. Expand the square norm in the left-hand side of (9.57), cancel out the term in $\int \rho(x)|x|^2/2$ on both sides, and then systematically integrate by parts and use the chain rule, in order to keep only terms of the forms $\int |\nabla \rho^\beta|^2$ and $\int \rho^\alpha$. By working out a homogeneity argument, it is possible to convince oneself that the family of inequalities (9.57) is a reformulation of a family of interpolation inequalities of the form

$$\|w\|_{L^p} \leq A_p \|\nabla w\|_{L^2}^{\theta} \|w\|_{L^{2(p-1)}}^{1-\theta} \tag{9.60}$$

when $0 < p < 2$ (here $p = 2m/(2m-1)$), and

$$\|w\|_{L^p} \leq A_p \|\nabla w\|_{L^2}^{\theta} \|w\|_{L^{\frac{p}{2}+1}}^{1-\theta} \tag{9.61}$$

when $2 < p \leq (2n)/(n-2)$ (here $p = 2/(2m-1)$). In these formulas, θ is an exponent in $[0,1]$, which is easily guessed by a homogeneity argument.

These inequalities belong to the family of **Gagliardo-Nirenberg inequalities**, which has been studied for some time in functional analysis. For $p = 2$, one recovers the Stam-Gross logarithmic Sobolev inequality as a particular case. For $p = (2n)/(n-2)$, one obtains the usual Sobolev inequality for functions in $H^1 = W^{1,2}(\mathbb{R}^n)$. Note that the non-existence of the

Barenblatt-Pattle profile in the case $n = 2$, $m = 1/2$ is related to the fact that $H^1(\mathbb{R}^2)$ is not imbedded in $L^\infty(\mathbb{R}^2)$.

Now, the truly amazing fact observed by Del Pino and Dolbeault is that these inequalities, which we obtained from McCann's displacement convexity, *are all optimal*! In fact, this is the way Del Pino and Dolbeault prove them: by a direct attack on the problem of optimal constants in (9.60)–(9.61). These same Gagliardo-Nirenberg inequalities were studied by optimal transportation methods more directly in [**101**], as we mentioned in Section 6.4. By the way, one could have guessed this connection by noticing the similarity between the Barenblatt-Pattle profile (9.58) and the minimizers in Sobolev's inequality (6.30).

Let us conclude with further results on the asymptotic behavior for porous medium equations. The methods described above, based on entropy production rates and optimal transportation, allowed specialists to solve this problem with optimal rates, leaving however two limitations on the exponents. The first was $m > n/(n+2)$, which is the condition for the Barenblatt profile to have finite second-order moments; and the second one, more fundamental, was $m \geq 1 - 1/n$, which is the condition for displacement convexity of the energy. On the other hand, a theory for porous medium equations with finite mass can be developed under less stringent assumptions: it suffices that $m > 1 - 2/n$, and this is the natural condition one would like to be able to treat.

As shown by Lederman and Markowich [**178**], the restriction $m > n/(n+2)$ can be removed if the initial datum (and as a consequence the whole time-dependent solution) is close enough to the Barenblatt profile as $|x| \to \infty$. But a more spectacular *coup de théâtre* occurred when Carrillo and Vázquez [**87**] discovered that the range $1 - 2/n < m < 1 - 1/n$ could be treated with the help of the Aronson-Bénilan inequalities [**23**], a family of famous estimates in the theory of porous medium equations. To state things in a nutshell, solutions of the porous medium equation $\partial\rho/\partial t = \Delta\rho^m$ are immediately regularized in a particular sense: $(m-1)^{-1}\Delta(\rho^{m-1})$ at time t can be uniformly bounded from below by an explicit negative constant, as soon as t is positive. This makes it possible to extract some uniform convexity property of the energy *along solutions of the partial differential equation* (in its rescaled version), and is sufficient to establish an entropy-entropy production inequality for positive times. On the whole, this provides a beautiful example in which estimates coming from the classical theory of porous medium equations are combined with the more exotic tools of entropy production estimates and optimal transportation, yielding in the end sharp decay results.

9.6. Nonlinear generalizations: interaction energy

In this last section we give a brief account on further work on the trend to equilibrium for equation (9.4). For simplicity we consider only the case where $U(s) = s \log s$; so the equation that we study is

$$\frac{\partial \rho}{\partial t} = \nabla \cdot \Big[\sigma \nabla \rho + \rho \nabla V + \rho \nabla (W * \rho)\Big], \tag{9.62}$$

where $\sigma \geq 0$, $D^2 V \geq 0$, $D^2 W \geq 0$, and W is symmetric. In this case, ρ_∞ in general is not explicit, which may at first sight cast some doubt on the possibility of getting somewhat sharp results.

The original motivations for this study were the models of granular flows discussed in Benedetto, Caglioti, Carrillo and Pulvirenti [**41, 40**]. We summarize below some results obtained by the author in collaboration with Carrillo and McCann [**85**]. As before, we shall use the notation

$$F(\rho) = \sigma \int_{\mathbb{R}^n} \rho \log \rho + \int_{\mathbb{R}^n} V(x)\, d\rho(x) + \frac{1}{2} \int_{\mathbb{R}^n \times \mathbb{R}^n} W(x-y)\, d\rho(x)\, d\rho(y),$$

$$D(\rho) = \int |\xi|^2 \, d\rho, \qquad \xi = -\nabla(\log \rho + V + \rho * W).$$

Under adequate assumptions on the growth and smoothness of V and W, and adequate moment assumptions on the initial datum ρ_0, one can construct well-behaved solutions of (9.62) satisfying the entropy production equality

$$\frac{d}{dt} F(\rho_t) = -D(\rho_t)$$

for $t > 0$. Here we skip these conditions and refer the reader to [**85**] for precise statements.

The first main result states that *a uniformly convex confinement potential implies an exponential convergence.* More precisely, if $D^2 V \geq \lambda I_n$ $(\lambda > 0)$, then one has the functional inequalities

$$D(\rho) \geq 2\lambda F(\rho|\rho_\infty), \qquad W_2(\rho, \rho_\infty) \leq \sqrt{\frac{2F(\rho|\rho_\infty)}{\lambda}}, \tag{9.63}$$

together with the convergence estimates

$$F(\rho_t|\rho_\infty) = O(e^{-2\lambda t}), \qquad W_2(\rho_t, \rho_\infty) = O(e^{-\lambda t}), \tag{9.64}$$

with constants depending only on $W_2(\rho_0, \rho_\infty)^2$.

In the sequel, we shall assume that V is not confining at all, and we shall even choose $V = 0$. This implies

$$\frac{d}{dt} \int x \, d\rho_t(x) = 0.$$

Thanks to this conservation law, it is possible to exploit the convexity of W in an efficient way. Thus the second rule states that *a uniformly convex interaction potential drives the system to equilibrium exponentially, if the center of mass is preserved in time.* More precisely, if $V = 0$ and $D^2W \geq \lambda I_n$ ($\lambda > 0$), then formulas (9.63), (9.64) still hold true. The condition on the center of mass is related to a physically significant feature of the interaction energy $\mathcal{W}$: it is invariant under translation of the probability density.

By putting together some of the elements in Chapter 5 and in the present chapter, the reader should now be able to establish these results by himself/herself!

Now, it is interesting to investigate what happens in the situation when W is not uniformly convex close to 0. The motivation for this study comes from the equations for granular media, in which the interaction potential is cubic, and therefore not uniformly convex close to $z = 0$. This loss of uniformity may result in a deterioration in the rate of convergence, which in general will be polynomial. So, our third rule states that *if W is strictly convex, in the sense that $D^2W(z) \geq K \min(|z|^\alpha, 1)$ for some positive constants K, α, then the system goes to equilibrium no slower than with an inverse polynomial rate.* More precisely, one has the functional inequalities

$$D(\rho) \geq \frac{\lambda}{C} F(\rho|\rho_\infty)^\beta, \qquad W_2(\rho, \rho_\infty) \leq CF(\rho|\rho_\infty)^\gamma,$$

and the convergence estimates

$$F(\rho_t|\rho_\infty) = O(t^{-1/(\beta-1)}), \qquad W_2(\rho_t, \rho_\infty) = O(t^{-\gamma/(\beta-1)})$$

for some positive constants C, β, γ.

Finally, the last rule is the most interesting; in fact it is quite surprising in the context of our general picture about uniform displacement convexity. It shows that the combined effect of the internal energy and of the interaction energy, neither of which is uniformly displacement convex, may result in an exponential trend to equilibrium.

Actually, *if $\sigma > 0$, and W is strictly convex in the same sense as in the above rule, then the trend to equilibrium is exponential*: formulas (9.63) and (9.64) still hold true for some λ which depends on ρ_0 only through an upper bound on $F(\rho_0)$. More precisely, if the center of mass is fixed, one can choose

$$\lambda = \inf_{t \geq 0} \frac{K}{8} \inf_{y,z \in \mathbb{R}^n} \int \min\big(|x-y|^\gamma, |x-z|^\gamma\big) \rho_t(x)\, dx.$$

If one recalls that $F(\rho_t) \leq F(\rho_0)$, then one can bound the constant λ above in terms of $F(\rho_0)$ and σ, as soon as $\sigma > 0$. The idea is that λ can be close to 0 only if ρ_t is concentrated on the union of two small balls, for some $t \geq 0$;

but this would not be compatible with the fact that $\int U(\rho_t)$ is uniformly bounded.

The last rule can be interpreted as follows: the diffusion forces the solution to be somewhat *spread*, and this is sufficient to make the interaction energy act much more efficiently to drive the system to equilibrium. Roughly speaking, two particles which are very close should rather interact indirectly, by first interacting with a third particle which would be somewhat far from both. This last rule applies even when there is also a confinement potential V: thanks to the spreading again, it is possible to show that possible moves of the center of mass do not affect the estimates too much.

It is now time to conclude this survey. The above-mentioned developments show how, at the time of the first edition of this book, optimal transportation had started to be blended into a mix of other mathematical fields involving geometric and probabilistic functional inequalities, and qualitative properties of partial differential equations. It was actually becoming part of the standard toolbox of analysts and mathematical physicists. Since then this picture has been refined and generalized, and optimal transportation would even make its way in some reinterpretations of Perelman's famous proof of the Poincaré conjecture... We refrain from saying more here, and refer once again to [**267**] for doomed but sincere attempts of exhaustively tracking the rich developments of this theory.

Problems

In this chapter the reader will find a number of problems of variable difficulty. Needless to say, their selection has been strongly influenced by the personal tastes and areas of competence of the author. Many of these problems have been inspired by research papers from the nineties. They have been tentatively ordered according to their area of applications; more often than not they require the use of notions taken from several chapters. Some hints are given at the end of each problem. Problems 1 to 3 concern various situations dealing with the geometry in $\mathbb{R}^2$. They will be an occasion to recall all the basic definitions and theorems from the first chapters. Problems 4 to 6 dig a little bit into the properties of spaces of probability measures, when equipped with a Wasserstein distance. Problems 7 to 13 consider various situations in which the Monge-Kantorovich problem and partial differential equations are mixed together; in Problems 11 and 13 an additional emphasis is laid on functional inequalities. Finally, the last three problems are inspired by statistical mechanics, an area in which mass transportation techniques seem to be particularly useful.

List of Problems

Problem 1. MK2 in Euclidean space commutes with translation

Here "MK2" stands for the Monge-Kantorovich problem with a quadratic transportation cost. This problem is adapted from a remark by Brenier, according to which MK2 is left invariant (in a sense which the reader should soon be able to make precise) by the action of the group of translations in $\mathbb{R}^n$. An interesting consequence will be deduced. This problem does not use any concepts beyond the definition of the mass transportation problem, as presented in the Introduction.

We consider the quadratic cost $c(x,y) = |x-y|^2$ in $\mathbb{R}^n$. When π is a transference plan between two probability measures μ and ν, we write $I[\pi] = \int |x-y|^2\, d\pi(x,y)$. For given $a \in \mathbb{R}^n$ we denote by τ_a the translation $x \longmapsto x+a$. In particular, $(\tau_a \times \mathrm{Id})(x,y) = (x+a, y)$.

1. Let μ, ν be two probability measures in $P_2(\mathbb{R}^n)$. Let $\pi \in \Pi(\mu,\nu)$; compute $I[(\tau_a \times \mathrm{Id})\#\pi)]$ in terms of $I[\pi]$. Use the result to justify the claim that optimal transference plans are left invariant by the action of translations in $\mathbb{R}^n$. We will now show an interesting geometrical consequence.

2. Find vectors x_1, x_2, y_1, y_2 in $\mathbb{R}^2$, with $x_2 = y_1$, such that the optimal transference plan between $\mu = \frac{1}{2}(\delta_{x_1} + \delta_{x_2})$ onto $\nu = \frac{1}{2}(\delta_{y_1} + \delta_{y_2})$ is given (with obvious abuse of terminology) by the transport of x_1 onto y_1, and of x_2 onto y_2.

3. Combine 1 and 2 to construct, without any calculation, four points x_1, x_2, y_1, y_2 in $\mathbb{R}^2$ for which the lines of optimal transportation (the straight lines joining initial and final position of points), in the transport process from $\frac{1}{2}(\delta_{x_1} + \delta_{x_2})$ onto $\frac{1}{2}(\delta_{y_1} + \delta_{y_2})$, do cross.

4. In particular, we conclude that transportation lines for the Monge-Kantorovich problem with cost $c(x,y) = |x-y|^2$ can cross. Is the same possible for the cost $c(x,y) = |x-y|$? (The answer was already known to Monge!)

Hints: *Question* 1: $I[(\tau_a \times \mathrm{Id})\#\pi] = I[\pi] + |a|^2 + 2\langle a, \int x\, d\mu(x) - \int y\, d\nu(y)\rangle$. So π is optimal in the transportation problem from μ to ν if and only if $(\tau_a \times \mathrm{Id})$ is in the transportation problem from $\tau_a\#\mu$ to ν. *Question* 4: What does c-cyclic monotonicity tell us here?

Problem 2. Transporting circles

This problem is a simple application of basic definitions and results in Chapters 2 and 7.

Let $C_{x,r}$ stand for the circle of center x and radius r in $\mathbb{R}^2$. Let $\Pi_{x,r}$ stand for the uniform measure on $C_{x,r}$, and let W_p be the Wasserstein distance of order p, for some $p \in [1, +\infty)$.

1. Give an upper bound on $W_p(\Pi_{x,r}, \Pi_{y,s})$ in terms of x, y, r, s. To be meaningful, this bound must go to 0 as $y \to x$, $s \to r$.

2. Give the exact value of $W_2(\Pi_{x,r}, \Pi_{y,s})$.

3. What about transporting ellipses?

Hint: In case of trouble, look back to Section 7.5.

Problem 3. Estimates of Grad's number

This problem is taken from the research paper [**119**]. The material of Chapters 2, 3 and 7 should be useful here, together with Theorem 5.34.

Let Ω be a smooth bounded open set in $\mathbb{R}^d$. Its Grad number $G(\Omega)$ is a geometric quantity which is useful to measure how close Ω is to, or far from, admitting an axis of symmetry. To introduce it, we shall need some definitions.

We define $H^1(\Omega; \mathbb{R}^d)$ to be the space of all vector fields $u = (u_i)_{1 \le i \le d}$ in $L^2(\Omega; \mathbb{R}^d)$ such that $\partial u_i / \partial x_j \in L^2(\Omega)$ for all indices i and j. Whenever $u \in H^1(\Omega; \mathbb{R}^d)$, we denote by ∇u its Jacobian matrix, and by $\nabla \cdot u$ its divergence:

$$(\nabla u)_{ij} = \frac{\partial u_i}{\partial x_j}, \qquad \nabla \cdot u = \mathrm{tr}\,(\nabla u).$$

We denote by $\nabla^s u$ and $\nabla^a u$ the symmetric and antisymmetric parts of ∇u, respectively:

$$(\nabla^s u)_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \qquad (\nabla^a u)_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right).$$

Whenever $\Sigma \in A_d(\mathbb{R})$ is an antisymmetric $d \times d$ matrix, we denote by $V_\Sigma(\Omega)$ the set of all vector fields $v \in H^1(\Omega; \mathbb{R}^d)$ satisfying

$$\begin{cases} \nabla \cdot v \equiv 0, \quad \nabla^a v \equiv \Sigma \quad \text{in } \Omega, \\ \\ v \cdot n = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{10.1}$$

where n stands for the unit outer normal to Ω, defined on $\partial\Omega$. It is possible to show that $V_\Sigma(\Omega)$ is not empty and that all elements in $V_\Sigma(\Omega)$ are smooth on $\overline{\Omega}$ (in the sequel, C^1 regularity will be sufficient). In fact, V_Σ contains exactly one element if Ω is simply connected.

If M is a matrix-valued application defined on Ω, we naturally define $\|M\|_{L^2(\Omega)}$ to be the L^2 norm of the mapping $x \longmapsto \|M(x)\|_{HS} = [\operatorname{tr}(MM^T)]^{1/2}$. We further define $UA_d(\mathbb{R})$ to be the set of all antisymmetric $d \times d$ real matrices Σ such that $\|\Sigma\|_{HS} = 1$. Now, **Grad's number** is defined as

$$G(\Omega) = \inf_{\Sigma \in UA_d(\mathbb{R})} \inf_{v \in V_\Sigma(\Omega)} \|\nabla^s v\|^2_{L^2(\Omega)}.$$

The quantity $G(\Omega)$ proves very useful in the study of certain geometrical inequalities, see [**119**]. In this problem we shall establish the following lower bound for $G(\Omega)$ in dimension 2. We use the notation $\mathcal{L}_\Omega$ for the Lebesgue measure restricted to Ω, and $\langle f \rangle_\Omega$ for the average of a function f on Ω, i.e. $|\Omega|^{-1} \int_\Omega f$.

Theorem 10.1 (A lower bound for Grad's number). *Let Ω be a smooth bounded open set in $\mathbb{R}^2$, and let $\mathcal{L}_\Omega$ be the Lebesgue measure on Ω. Define the center of mass g of Ω by*

$$g = \frac{1}{|\Omega|} \int_\Omega x \, dx.$$

For each $\theta \in [0, 2\pi)$, define Ω_θ as the image of Ω under rotation through the angle θ around g, and define the measure $\mathcal{L}_\Omega^{\text{sym}}$ by symmetrization of $\mathcal{L}_\Omega$:

$$\mathcal{L}_\Omega^{\text{sym}} = \frac{1}{2\pi} \int_0^{2\pi} \mathcal{L}_{\Omega_\theta} \, d\theta.$$

Then there exists an explicit numerical constant K such that

$$G(\Omega) \geq \frac{K}{P(\Omega)} W_2 \left(\frac{1}{|\Omega|} \mathcal{L}_\Omega, \ \frac{1}{|\Omega|} \mathcal{L}_\Omega^{\text{sym}} \right)^2, \tag{10.2}$$

where $P(\Omega)$ stands for the Poincaré constant of Ω, i.e. the smallest admissible constant in the functional inequality

$$\|f - \langle f \rangle_\Omega\|^2_{L^2(\Omega)} \leq P(\Omega) \, \|\nabla f\|^2_{L^2(\Omega)}.$$

Remark 10.2. A corollary of the theorem is that $G(\Omega) = 0$ if and only if Ω has circular symmetry, i.e. if it is left invariant under rotation through an arbitrary angle around its center of mass; but the interest of the theorem is precisely that it gives a more precise lower bound. Note that $\mathcal{L}_\Omega^{\mathrm{sym}}$ is not the Lebesgue measure on the symmetrized set $\bigcup \Omega_\theta$!

Let us go on with the proof of Theorem 10.1. In the beginning we do not need to assume $d = 2$.

1. Let $\Sigma \in UA_d(\mathbb{R})$, and let $v \in V_\Sigma(\Omega)$. As we mentioned above, v can be extended into a C^1 map on $\overline{\Omega}$. For $t \in \mathbb{R}$ define the exponential map e^{tv} by the identity

$$\frac{d}{dt} e^{tv}(x) = v(e^{tv}(x)),$$

together with the initial condition $e^0 = \mathrm{Id}$. How can one express the connection between v and e^{tv}, in the language of fluid mechanics? Let $x \in \partial\Omega$; show that $e^{tv}(x) \in \partial\Omega$ for all t. Invoking theorems of uniqueness of solutions of ordinary differential equations, show that

$$x \in \Omega \Longrightarrow [\forall t \in \mathbb{R},\ e^{tv}(x) \in \Omega].$$

Conclude that e^{tv} maps Ω into itself. Then show that e^{tv} actually is a C^1 diffeomorphism from Ω onto itself. Finally, show that $e^{tv}\#\mathcal{L}_\Omega = \mathcal{L}_\Omega$.

2. Let $b \in \mathbb{R}^d$. Define the affine map R on Ω by

$$R(x) = \Sigma x + b.$$

Such an affine map (with antisymmetric linear part) is called a **rigid body motion**. Introduce the exponential map e^{tR} in the same way as above:

$$\frac{d}{dt} e^{tR}(x) = R(e^{tR}(x)).$$

Then show that $e^{tR}\#\mathcal{L}_\Omega = \mathcal{L}_{e^{tR}(\Omega)}$. Finally, show that

$$\frac{1}{\Omega} \int_\Omega |e^{tv}(x) - e^{tR}(x)|^2 \, dx \geq W_2\left(\frac{1}{\Omega}\mathcal{L}_\Omega, \frac{1}{\Omega}\mathcal{L}_{e^{tR}(\Omega)}\right)^2. \tag{10.3}$$

3. Using a Gronwall lemma, establish the inequality

$$|e^{tv}(x) - e^{tR}(x)| \leq e^t \int_0^t |v(e^{sv}(x)) - R(e^{sv}(x))| \, ds.$$

4. Conclude that for all $T > 0$ there exists a constant $C(T)$ such that

$$\frac{1}{T} \int_0^T \int_\Omega |e^{tv}(x) - e^{tR}(x)|^2 \, dx \, dt \leq C(T) \int_0^T \int_\Omega |v(e^{tv}(x)) - R(e^{tv}(x))|^2 \, dx \, dt.$$

5. Deduce that for all $T > 0$ there exists a constant $C'(T)$ such that

$$\frac{1}{T}\int_0^T \int_\Omega |e^{tv}(x) - e^{tR}(x)|^2 \, dx \, dt \le C'(T) \int_\Omega |v(x) - R(x)|^2 \, dx.$$

6. Show that for a suitable choice of b,

$$\begin{aligned}\int_\Omega |v(x) - R(x)|^2 \, dx &\le P(\Omega) \int_\Omega \|\nabla v(x) - \Sigma\|_{HS}^2 \, dx \\ &= P(\Omega) \int_\Omega \|\nabla^s v(x)\|_{HS}^2 \, dx.\end{aligned}$$

7. Let R_g be defined by $R_g(x) = \Sigma(x - g) + g$. In words, R_g is the rigid body motion having the same linear part as R, which preserves the center of mass g of Ω. Show that, for all t,

$$\int_\Omega |e^{tv}(x) - e^{tR}(x)|^2 \, dx \ge \int_\Omega |e^{tv}(x) - e^{tR_g}(x)|^2 \, dx.$$

8. Put everything together and show that for some constant $K(T) > 0$ one has

$$G(\Omega) \ge \frac{K(T)}{2P(\Omega)} \inf_{R \in \mathcal{R}_{1;g}} \left(\frac{1}{T} \int_0^T W_2\left(\frac{1}{\Omega}\mathcal{L}_\Omega, \frac{1}{\Omega}\mathcal{L}_{e^{tR}\Omega}\right)^2 dt \right),$$

where $\mathcal{R}_{1;g}$ is the set of all rigid motions R whose linear part Σ is an element of $UA_d(\mathbb{R})$, such that $R(g) = g$.

9. Now set $d = 2$ and conclude the proof of Theorem 10.1.

10. We denote by $B(0, r)$ the ball of radius r centered at 0. Let $(\Omega_\varepsilon)_{\varepsilon > 0}$ be a family of open sets in $\mathbb{R}^2$, obtained by deformation of $B(0, 1)$ in such a way that

$$B(0, 1 - \varepsilon) \subset \Omega_\varepsilon \subset B(0, 1 + \varepsilon),$$
$$\big|\Omega_\varepsilon \setminus B(0, 1 - \varepsilon)\big| \ge K\varepsilon, \qquad \big|B(0, 1 + \varepsilon) \setminus \Omega_\varepsilon\big| \ge K\varepsilon,$$

for some $K > 0$ independent of ε. Show that there exists $K' > 0$ such that

$$G(\Omega_\varepsilon) \ge K'\varepsilon^2.$$

Hints: *Question* 0: Use the continuity properties of the Wasserstein distance. *Question* 1: Notice that e^{tv} induces a field of tangent vectors on $\partial\Omega$, viewed as a manifold. You can show that $e^{tv} \circ e^{-tv} = \mathrm{Id}$. In case of difficulty, consult Section 3.2. *Question* 3: Recall that $\|\Sigma\|_{HS} = 1$. *Question* 6: Reduce to the case where $v - R$ has zero mean. *Question* 7: Note that $e^{tR} - e^{tR_g}$

is a constant vector, and show that $\int e^{tv}(x)\,dx = \int e^{tR_g}(x)\,dx = g$. Note the resemblance to Question 1 of Problem 1. *Question* 9: Use the convexity of W_2^2. *Question* 10: Recall subsection 7.1.2 and Corollary 1.16. Note that a less careful argument would yield a weaker bound in ε^4.

The very same arguments leading to the proof of Theorem 10.1 also imply the following bound on $G(\Omega)$ in dimension $d = 3$. Whenever $\sigma \in S^2$, define by $\mathcal{L}_\Omega^{\mathrm{sym};\sigma}$ the measure obtained by symmetrizing $\mathcal{L}_\Omega$ around the axis directed by σ, passing through g. Then

$$G(\Omega) \geq \frac{K}{P(\Omega)} \inf_{\sigma \in S^2} W_2 \left(\frac{1}{|\Omega|}\mathcal{L}_\Omega, \frac{1}{|\Omega|}\mathcal{L}_\Omega^{\mathrm{sym};\sigma} \right)^2 .$$

In particular, $G(\Omega) = 0$ if and only if Ω is axisymmetric, i.e. left invariant under rotations through an arbitrary angle around some axis (which necessarily should contain g, of course).

Problem 4. Approximation lemmas for transference plans

This problem answers some natural questions about transference plans. It only requires the material in Chapter 7.

1. Let X and Y be Polish spaces; show that $X \times Y$, equipped with its natural topology of product space, is also a Polish space. Construct a Monge-Kantorovich distance metrizing the weak topology on $P(X \times Y)$.

2. Now let $\mu \in P(X)$ and $\nu \in P(Y)$, and let $\pi \in \Pi(\mu, \nu)$. Let $(\mu_n)_{n\in\mathbb{N}}$ and $(\nu_n)_{n\in\mathbb{N}}$ be sequences of probability measures converging to μ and ν respectively, as $n \to \infty$. Show that there exists a sequence $(\pi_n)_{n\in\mathbb{N}}$ of transference plans such that

$$\pi_n \in \Pi(\mu_n, \nu_n) \qquad \text{and} \qquad \pi_n \xrightarrow[n\to\infty]{} \pi.$$

3. Let μ, ν and π be as above. Since X and Y are Polish spaces, there exist sequences $(x_k)_{k\in\mathbb{N}}$ and $(y_\ell)_{\ell\in\mathbb{N}}$, dense in X and Y respectively. Show that there exist sequences of discrete probability measures

$$\mu_n = \sum_{k=1}^{N_n} \alpha_{n,k}\, \delta_{x_k}, \qquad \nu_n = \sum_{\ell=1}^{N_n} \beta_{n,\ell}\, \delta_{y_\ell}, \qquad \pi_n = \sum_{k=1}^{N_n} \sum_{\ell=1}^{N_n} \gamma_{n,k,\ell}\, \delta_{(x_k, y_\ell)}$$

such that

$$\mu_n \longrightarrow \mu, \qquad \nu_n \longrightarrow \nu, \qquad \pi_n \longrightarrow \pi.$$

4. Let μ, ν be probability measures on (say) $X = Y = [0,1]$, and let $\pi \in \Pi(\mu,\nu)$. Construct (as explicitly as possible) sequences μ_n and T_n such that $\pi_n = (\mathrm{Id} \times T_n)\#\mu_n$ converges weakly to π. In words, at least if we leave some freedom to the marginal distributions, transference plans for the Monge problem form a dense subset of transference plans for the Kantorovich problem. Estimate the distance from π_n to π in some appropriate distance.

Remark 10.3. In fact it is not always possible to require the second marginal of π_n be ν; but this is true if μ, ν have no atoms.

Hints: *Question* 1: Recall the remarks following Theorem 7.12. *Question* 2: Consult Theorem 7.12 and apply Lemma 7.6 twice. *Question* 3: For the construction of μ_n and ν_n, one can adapt the strategy used in Problem 6 below. *Question* 4: Here is a possible strategy, explained in an informal manner. First use an approximation argument to reduce to the case in which $Z = [0,1] \times [0,1]$ is divided into a grid of small squares, and the density of π over each square is a constant, integer multiple of $1/N$ for some $N \in \mathbb{N}$. Then, separate each column of Z into a large number of sub-columns, and require that on each of these sub-columns, T_n lies in a fixed row. Arrange that within a given column C, those rows R such that $\pi[R \cap C]$ is large are visited many times by T_n. Even with those indications, the precise construction and the distance estimates should require some effort from the reader!

Problem 5. Joint information lemmas

This problem is elementary and only needs basic notions from Chapter 7. It also deals with the functional encountered in Chapter 9.

Whenever μ is a probability measure on $\mathbb{R}^n$, absolutely continuous with respect to Lebesgue measure, with density f, such that $(f \log f)_- \in L^1(\mathbb{R}^n)$, define

$$H(\mu) = \int_{\mathbb{R}^n} f \log f \, dx, \qquad I(\mu) = 4\int_{\mathbb{R}^n} |\nabla \sqrt{f}|^2 \, dx.$$

The usual "joint information lemma" in information theory states the following (see for instance [**102**]). Let μ be a probability measure on $\mathbb{R}^n = \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$, with marginals μ_1 and μ_2 on $\mathbb{R}^{n_1}$ and $\mathbb{R}^{n_2}$ respectively. Assume that $H(\mu_1)$, $H(\mu_2)$ and $H(\mu)$ are well-defined. Then,

$$H(\mu) \geq H(\mu_1) + H(\mu_2); \tag{10.4}$$

moreover, if $H(\mu) < +\infty$, then equality in the above inequality holds if and only if $\mu = \mu_1 \otimes \mu_2$.

The usual interpretation, in the language of information theory, is crystal clear: knowing the joint law of a pair of random variables (U_1, U_2) brings more information than knowing just the law of U_1 and the law of U_2 — except if they are independent!

1. Prove inequality (10.4). Note that this property in fact does not need a Euclidean setting: it still holds true if the reference Lebesgue measures on $\mathbb{R}^{n_1}$, $\mathbb{R}^{n_2}$ and $\mathbb{R}^n$ are replaced by reference measures σ_1, σ_2 and σ such that $\sigma = \sigma_1 \otimes \sigma_2$.

2. Prove a similar inequality with I in place of H. This is a particular case of some inequalities proven by Carlen [**83**].

3. Prove a similar inequality for the Wasserstein distance of order 2. Namely, when μ, ν are probability measures on $\mathbb{R}^n = \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ with respective marginals μ_1 and μ_2, ν_1 and ν_2, then

$$W_2(\mu,\nu)^2 \geq W_2(\mu_1,\nu_1)^2 + W_2(\mu_2,\nu_2)^2.$$

Generalize this to the case when μ and ν are probability measures on $\mathbb{R}^n = \mathbb{R}^{n_1} \times \cdots \mathbb{R}^{n_k}$ with respective marginals $\mu_1, \ldots, \mu_k$ and $\nu_1, \ldots, \nu_k$. Under which assumptions does this lemma survive in a more general setting than just Euclidean space?

Hints: *Question* 1: Note that $H(\mu) - H(\mu_1) - H(\mu_2) = H(\mu|\mu_1 \otimes \mu_2)$ and recall the exercise on page 282. *Question* 2: When f is smooth, then

$$I(\mu) = \int f |\nabla(\log f)|^2 \, dx.$$

This form makes it not too difficult to check that (for smooth densities)

$$I(\mu) - I(\mu_1) - I(\mu_2) = I(\mu|\mu_1 \otimes \mu_2),$$

where $I(\cdot|\cdot)$ is the relative Fisher information defined in (9.25). If f is not smooth, one can use an approximation argument.

Problem 6. Metric entropy of a space of probability measures

This problem is inspired by Exercise 6.2.19 in [**116**]. It only requires a good understanding of the definition of optimal transportation.

Let X be a Polish space equipped with a distance d, and let $p \geq 1$ and $r > 0$. Let $M_p(X, r)$ be the minimal number of balls of radius r in the metric W_p which one needs to cover $P(X)$. Our goal is to establish an upper bound on $M_p(X, r)$ in terms of (i) the **diameter** of X, defined by the

formula $D(X) = \sup\{d(x,y);\ x \in X, y \in Y\}$; and (ii) the **metric entropy** of X, i.e. the function $r \mapsto m(X,r)$, where $m(X,r)$ is defined as the minimal number of balls of radius r needed to cover X.

1. Why is it true that $M_p(X,r) \geq m(X,r)$?

2. Let $r_1 \in (0,1/2)$, to be chosen later. Let $\{B(x_j, r_1)\}_{1\leq j\leq J}$ be a family of balls of radius r_1 covering X, with $J = m(X, r_1)$. Of course all x_j's are distinct (why?). Introduce the family of balls $\mathcal{B}(\mu, r)_{\mu\in\mathcal{C}}$ in $P(X)$, having radius r_1 in the Wasserstein distance W_p, with centers belonging to the set

$$\mathcal{C} = \left\{ \mu = \sum_{j=1}^{J} \alpha_j \delta_{x_j}, \qquad \alpha = (\alpha_j)_{1\leq j\leq J} \in A \right\},$$

where A is the set of all J-tuples of numbers $\alpha_j \geq 0$ such that $\sum \alpha_j = 1$, and each α_j is an integer multiple of $1/K$ (the integer K will be chosen later). Show that $\mathcal{C}$ contains exactly

$$\begin{pmatrix} K+J-1 \\ K \end{pmatrix} = \frac{(K+J-1)!}{K!\,(J-1)!} \quad \text{elements.}$$

3. Define $K = [J/r_1^p] + 1$, where $[x]$ stands for the integer part of x. Check the following bound on the number of elements in $\mathcal{C}$:

$$\#\,C \leq \frac{(2K)^J}{J!} \leq \left(\frac{C}{r_1^p}\right)^J,$$

where C is a numerical constant.

4. Let μ be an arbitrary probability measure on X. Show that there exists a probability measure $\widetilde{\mu} = \sum \beta_j \delta_{x_j}$ such that $W_p(\mu, \widetilde{\mu}) \leq r_1$.

5. Show that there exists $\mu' \in \mathcal{C}$ such that

$$W_p(\widetilde{\mu}, \mu') \leq \left(\frac{J}{K}\right)^{\frac{1}{p}} D(X) \leq r_1 D(X).$$

6. Conclude that the family $\big(\mathcal{B}(\mu, 2r_1 D)\big)_{\mu\in\mathcal{C}}$ is a covering of $P(X)$, where $D = \max(1, D(X))$, and deduce that

$$M_p(X,r) \leq \left(\frac{C\,D}{r^p}\right)^{m\left(X, \frac{r}{2D}\right)},$$

for some numerical constant C.

7. As an example, consider the case when X is the Euclidean ball $B(0,R)$ of radius R in $\mathbb{R}^d$, and show that, for $R \geq r$,

$$m(B(0,R),r) \leq C\left(\frac{R}{r}\right)^d$$

for some constant C only depending on d. What does the bound in Question 6 become for $p=1$? Investigate other possible bounds by changing the choices of K and r_1.

Hints: *Question* 2: Show that this quantity is the number of ways of putting K indistinguishable objects into J distinguishable boxes. *Question* 3: Use for instance the inequality $(n/e)^n \leq Cn!$. *Question* 4: One possibility is to start by dividing the whole space X into "Voronoï cells" around the x_j's, i.e. the cell around x_j will contain all points in X which are strictly closer to x_j than to any other x_k (in case a point is equally distant from several x_j's, it can choose to go in whichever cell it likes). *Question* 6: Combine the results of questions 3 and 4.

Problem 7. A reflection theorem for the Monge-Ampère equation

This problem is taken from Caffarelli [**79**]. We shall see applications of it in Problem 12. It is based on the material in Chapters 2 and 4.

Let Ω be a convex open set in $\mathbb{R}^{n-1}$, and $Q=\overline{\Omega}\times[0,1]$. Define R to be the reflection across the hyperplane $H=\{x\in\mathbb{R}^n;\ x_n=0\}$, namely

$$R(x_1,\ldots,x_n)=(x_1,\ldots,x_{n-1},-x_n).$$

Let f and g be $C^{1,\alpha}$ positive probability densities on $\overline{Q}$ and let φ be convex on Q such that $\nabla\varphi\#f=g$. We assume that ∇f and ∇g are tangent to the boundary of Q at $x_n=0$, namely $Q\cap H$. Our goal is to show that $\nabla\varphi$ maps the interior of $Q\cap H$ into itself, in a $C^{2,\alpha}$ way.

1. Define $\overline{f}$ and $\overline{g}$ on $Q\cup R(Q)$ by reflecting f and g: for instance, $\overline{f}|_Q=f$ and $\overline{f}\circ R=\overline{f}$. What is the regularity of $\overline{f}$ and $\overline{g}$?

2. Let $\overline{\varphi}$ be a convex function on $Q\cup R(Q)$ such that $\nabla\overline{\varphi}\#\overline{f}=\overline{g}$ (where f and g are identified with the measures that they define). Show that $\nabla\overline{\varphi}\circ R=R\circ\nabla\overline{\varphi}$.

3. Let $T: Q\cup R(Q)\longmapsto Q\cup R(Q)$ be a monotone map (i.e. such that $\langle T(x)-T(y),\, x-y\rangle\geq 0$ for all x, y) satisfying $T\circ R=R\circ T$; show that $T(Q)\subset Q$. Deduce that the restriction $\widetilde{\varphi}$ of $\overline{\varphi}$ to Q satisfies $\nabla\widetilde{\varphi}\#f=g$, and therefore $\nabla\overline{\varphi}|_Q=\nabla\varphi$.

4. Using Caffarelli's regularity theory, show that $\nabla\overline{\varphi}$ is of class $C^{2,\alpha}$ in the interior of $Q \cup R(Q)$, and conclude.

5. Application: Let f and g be two positive, Hölder-continuous, 1-periodic functions on $\mathbb{R}^n$. Here 1-periodicity means that $f(x+e_i) = f(x)$ for any basis vector e_i. For simplicity, assume in addition that f and g are invariant under reflection around any hyperplane $(x_j = 0)$. Construct a classical solution of the Monge-Ampère equation

$$\det D^2\varphi(x) = \frac{f(x)}{g(\nabla\varphi(x))}$$

taking the form

$$\varphi(x) = \frac{|x|^2}{2} + \zeta(x),$$

where ζ is 1-periodic.

Hints: *Question* 1: $\overline{f}$ and $\overline{g}$ are still of class $C^{1,\alpha}$. *Question* 2: Use the uniqueness of the Brenier map, after noting that $R \circ \nabla\overline{\varphi} \circ R = \nabla(\overline{\varphi} \circ R)$. *Question* 4: Notice that the interior of $Q \cap H$ lies in the interior of $Q \cup R(Q)$. *Question* 5: First construct φ on the unit cube $[0,1]^n$, then show that $\nabla\varphi$ maps each face of the cube into itself, and extend φ to the whole of $\mathbb{R}^n$ by repeated reflections in all directions.

Problem 8. A nonlinear slowing down of convergence to equilibrium

This problem is taken from the research paper [**82**]. It is based on the material in Section 2.2, the characteristic methods set forth in Theorem 5.34, plus a little bit of Chapters 8 and 9. We shall see here how mass transportation tools enable us to solve some delicate problems in a straightforward way.

Our goal is the study of the asymptotic behavior of solutions to the partial differential equation

$$\frac{\partial f}{\partial t} = \frac{\partial}{\partial x}\left(f\left(f * \frac{\partial W}{\partial x}\right)\right) - \frac{\partial}{\partial x}(fx), \qquad t \geq 0, \quad x \in \mathbb{R},$$

where the unknown is a nonnegative time-dependent probability density $f_t(x) = f(t,x)$ on $\mathbb{R}$. Moreover,

$$W(x) = \frac{|x|^3}{3}.$$

As explained in [**41**, **82**], this system arises in the study of the asymptotic behavior of an oversimplified model of granular flow first suggested by the physicists McNamara and Young [**207**].

1. Check that

$$\mu_\infty = \frac{1}{2}(\delta_{-\frac{1}{2}} + \delta_{\frac{1}{2}})$$

is a stationary solution. Is this the only one?

2. Show that the energy

$$E(f) = \frac{1}{2}\int_{\mathbb{R}\times\mathbb{R}} W(x-y)\, f(x)f(y)\,dx\,dy - \int_{\mathbb{R}} f(x)\frac{|x|^2}{2}\,dx$$

acts as a Lyapunov functional for this system. Give an interpretation of the equation in terms of gradient flows. Why is it not obvious from the form of the functional that the equation will eventually converge to the minimum of the functional?

It was proven in [**41**] that μ_∞ is the unique minimizer of E on $P(\mathbb{R}^n)$, and that, if the initial datum f_0 is absolutely continuous with respect to Lebesgue measure and compactly supported, then the support of f_t remains within a fixed compact $K \subset \mathbb{R}$ for all times, and $f_t \longrightarrow \mu_\infty$ as $t \to \infty$. The purpose of this problem is to show that in fact the convergence is necessarily *very slow*. For simplicity we shall assume here that f_0 is strictly positive almost everywhere on some nontrivial interval around its median m_0 (which is therefore unique). Recall that that by definition $m \in \mathbb{R}$ is a **median** of the probability density f on $\mathbb{R}$ if

$$\int_{-\infty}^{m} f(x)\,dx = \frac{1}{2}, \qquad \int_{m}^{+\infty} f(x)\,dx = \frac{1}{2}.$$

3. Let $(f_t)_{t\geq 0}$ be a solution of the equation, compactly supported as a function of x. Show that $f_t = T_t\# f_0$, where T_t is a solution of

$$\frac{\partial}{\partial t}T_t = \xi_t \circ T_t, \qquad T_0 = \mathrm{Id}\,, \tag{10.5}$$

$$\xi_t(x) = x - \int_{\mathbb{R}} (x-y)|x-y|\,f_t(y)\,dy. \tag{10.6}$$

What is the regularity of ξ_t?

4. Show that T_t is always nondecreasing.

5. Show that $m_t = T_t(m_0)$ is the median of f_t, for all $t \geq 0$.

6. Compute $W_1(f_t, \mu_\infty)$, as an integral expression involving f_t and m_t.

7. Show that, for each $t \geq 0$, $d\xi_t(x)/dx$ is maximum at m_t.

8. Show that

$$1 - 2\int_{\mathbb{R}} f_t(x)|x - m_t|\, dx \leq 2W_1(f, \mu_\infty).$$

We are now ready to prove the slowness of the convergence. The physical intuition behind this fact is the following: if we look at the equation as a nonlinear transport equation, then the rate of variation of the density of particles is measured by the divergence of the velocity field, in our case $d\xi_t/dx$. In view of the form of the equilibrium, all the particles which are on the left of the median should go to the position $-1/2$, and all the particles on the right of the median should go to the position $+1/2$. In particular, the variation of the velocity field ξ_t has to be strong enough to separate particles which were very close initially. This can be true only if the time-integral of the maximum divergence of the velocity field is infinite. On the other hand, the velocity field associated (by the nonlinear coupling (10.6)) to the equilibrium state μ_∞ has zero divergence; therefore, by question 8, the divergence of ξ_t is close to 0 if f_t is close to equilibrium. This, combined with the fact that the time-integral of the divergence should diverge (*sic*), will imply that the convergence has to be slow. We shall implement these ideas in the sequel.

9. Let $\delta > 0$, small enough, to be chosen later on. Define

$$a_\pm = m_0 \pm \delta, \qquad a_\pm(t) = T_t(a_\pm).$$

Show that

$$\frac{d}{dt}[a_+(t) - a_-(t)] \leq 2\,[a_+(t) - a_-(t)]\, W_1(f_t, \mu_\infty).$$

10. From our assumption, there exists a function $\varepsilon(\delta)$ such that $\varepsilon(\delta) > 0$ if $\delta > 0$, and

$$\int_{a_-}^{m_0} f_0 \geq \varepsilon(\delta), \qquad \int_{m_0}^{a_+} f_0 \geq \varepsilon(\delta).$$

Show that these inequalities still hold true with $f_0, m_0, a_\pm$ replaced by $f_t, m_t, a_\pm(t)$ respectively. Deduce that

$$W_1(f_t, \mu_\infty) \geq \varepsilon(\delta)\Big[1 - (a_+(t) - a_-(t))\Big].$$

11. Show that

$$\exp\left(2\int_0^t W_1(f_s,\mu_\infty)\,ds\right) \geq \frac{1-\frac{1}{\varepsilon(\delta)}W_1(f_t,\mu_\infty)}{2\delta}$$

and conclude that

$$\int_0^{+\infty} W_1(f_t,\mu_\infty)\,dt = +\infty.$$

In particular, it is impossible that $W_1(f_t,\mu_\infty) = O(t^{-\kappa})$ for some $\kappa > 1$.

12. Assume that f_0 is bounded below by a positive number K close to m_0, and show that there exists $K' > 0$ such that for all $T > 0$,

$$\int_0^T W_1(f_t,\mu_\infty)\,dt \geq K' \log T.$$

Remark 10.4. The assumption of positivity around the median is in fact not necessary; neither is the assumption of absolute continuity with respect to Lebesgue measure: the result holds true as soon as f_0 is not a symmetric convex combination of two Dirac masses, see [**82**].

Hints: *Question* 1: This is not the only one; one might also think of combinations involving δ_0. *Question* 2: Note that the energy is not displacement convex! This is consistent with the fact that the set of equilibria is not invariant under displacement interpolation. *Question* 3: ξ_t is of class at least C^1 (to prove C^1 smoothness with respect to t, you may use the assumption of compact support). *Question* 4: Notice that T_0 is nondecreasing and use a continuity argument together with the Cauchy-Lipschitz theorem. *Question* 5: Use the definition of push-forward. *Question* 6: Since the problem is one-dimensional, one finds $\int_{-\infty}^{m_t} |x+1/2|\, f_t(x)\,dx + \int_{m_t}^{+\infty} |x-1/2|\, f_t(x)\,dx$. *Question* 7: Show that the derivative of $d\xi_t/dx$ is zero at m_t, and use a convexity argument to conclude that $d\xi_t/dx$ is nondecreasing for $x < m_t$, nonincreasing for $x > m_t$. *Question* 9: Use the characteristic equation, then apply Question 7 (together with the mean-value theorem), then Question 8. *Question* 10: Use the definition of push-forward and bound the optimal transportation cost explicitly from below. *Question* 11: Assume that the integral is convergent, and note that one can find arbitrary large times for which $W_1(f_t,\mu_\infty)$ is as small as desired; deduce that $\exp(2\int_0^{+\infty})$ has to be larger than $1/(2\delta)$, for all $\delta > 0$. *Question* 12: Choose $\varepsilon(\delta) = K\delta$, choose δ properly and apply a Gronwall lemma.

Problem 9. The semi-geostrophic system

This problem uses material from Chapters 3 and 8, together with the Lagrangian formalism and a little bit of Chapter 7. We first introduce briefly the semi-geostrophic system and then present the ideas suggested by various authors to rewrite it in an appealing way. Questions 1 to 4 are independent of the sequel.

The semi-geostrophic system arises in meteorology, for the study of atmospheric fronts, see Hoskins [**169, 170, 171**], Cullen and collaborators [**106, 107, 108, 109**] and references therein. Apparently this system first appeared in the forties, in a paper by Eliassen [**130**]. As a pedagogical reference for mathematicians we recommend Benamou's master's thesis [**37**]. To describe the semi-geostrophic system, it will be convenient to use the notation $U = (u, v, w)$ for the velocity field in $\mathbb{R}^3$, $X = (x, y, z)$ for the position variable. The starting point is the system of incompressible Boussinesq equations (a variant of Euler's system) within a frame rotating at angular velocity $\omega > 0$, taking into account the effects of gravity in the vertical direction, and assuming that the system is a slight perturbation of a hydrodystatic state of rest with uniform density ρ_0 and temperature θ_0:

$$\frac{D}{Dt}\begin{pmatrix} u \\ v \\ w \end{pmatrix} + \begin{pmatrix} \partial_x p & - & \omega v \\ \partial_y p & + & \omega u \\ \partial_z p & + & g\rho/\rho_0 \end{pmatrix} = 0. \tag{10.7}$$

Here D/Dt stands for the convective derivative,

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + U \cdot \nabla_X = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z},$$

ρ is a density perturbation and p a pressure perturbation. For simplicity we have taken into account the rotation of the Earth as if we were standing at the North Pole. To equation (10.7) should be added the incompressibility condition

$$\nabla_X \cdot U = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0, \tag{10.8}$$

the passive transport of temperature,

$$\frac{D\theta}{Dt} = 0, \tag{10.9}$$

and an equation of state linking the temperature perturbation θ with the density perturbation ρ: for instance,

$$\rho = -\alpha\theta, \tag{10.10}$$

where $\alpha > 0$ is given. Equations (10.7), (10.8), (10.9), (10.10) constitute a system of six scalar equations for the six unknowns u, v, w, ρ, θ and p.

If one is interested in the regime when ω is very large with respect to other velocity scales, various approximations are possible. The resulting equations are essentially two-dimensional. The most common approximation is the hydrostatic geostrophic approximation, in which the terms Du/Dt, Dv/Dt and Dw/Dt are neglected in the left-hand side of (10.7), so that

$$v = \frac{1}{\omega}\frac{\partial p}{\partial x}, \qquad u = -\frac{1}{\omega}\frac{\partial p}{\partial y}.$$

From the third equation we also have

$$\frac{\partial p}{\partial z} = -\frac{\rho g}{\rho_0} = \frac{\alpha \theta g}{\rho_0}.$$

By definition, the **geostrophic wind** is the velocity field which for a given pressure is predicted by the geostrophic approximation, namely

$$v_g = \frac{1}{\omega}\,\frac{\partial p}{\partial x}, \qquad u_g = -\frac{1}{\omega}\,\frac{\partial p}{\partial y}. \tag{10.11}$$

Thus, from (10.7) one has

$$v = v_g + \frac{D_t u}{\omega}, \qquad u = u_g - \frac{D_t v}{\omega}.$$

Re-injecting these values in (10.7), one finds that

$$v = v_g + \frac{D_t u_g}{\omega} - \frac{D_t D_t v}{\omega^2}, \qquad u = u_g - \frac{D_t v_g}{\omega} - \frac{D_t D_t u}{\omega^2}.$$

Neglecting terms of order ω^{-2} in this equation, one arrives at the **incompressible semi-geostrophic system**:

$$\begin{cases} \dfrac{Du_g}{Dt} - \omega v + \dfrac{\partial p}{\partial x} = 0, \\[2ex] \dfrac{Dv_g}{Dt} + \omega u + \dfrac{\partial p}{\partial y} = 0, \end{cases} \tag{10.12}$$

which is coupled with (10.11). This system is quite tricky to study, and for the moment it is not clear that it should be well-posed, even from the formal point of view. One goal of this exercise is to get an idea of why it should indeed be well-posed. Surprisingly, the solution will lead us to optimal mass transportation.

In the sequel, we will assume that physical units have been scaled in such a way that $\omega = 1$ and $\theta = \partial p/\partial z$. Moreover, the velocity field lives on a bounded domain Ω, whose volume is normalized to unity, with a tangency condition on the boundary. The reader should not worry about regularity issues (which are quite delicate), but only try to understand the formal structure of the whole.

1. Introduce the "semi-geostrophic coordinates"

$$X_g = (x_g, y_g, z_g) \equiv (v_g + x, y - u_g, \theta)$$

and the "geopotential"

$$P_g(X) = p(X) + \frac{1}{2}(x^2 + y^2),$$

and show that

$$X_g = \nabla P_g(X).$$

Then rewrite the semi-geostrophic equations in the form

$$\frac{DX_g}{Dt} = J \cdot (X_g - X),$$

where the matrix J is

$$J = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

At this stage, it is still unclear why the system should be well-posed, because the convective derivative depends on the velocity field itself!

2. Next go to Lagrangian formalism and define $L(t, X_0)$ as the position at time t of a particle which started at time 0 from position X_0 and was subsequently advected by the velocity field U. Define the time-dependent mapping m by

$$m(t, X_0) = -\nabla P_g(t, L(t, X_0)),$$

and check that the semi-geostrophic equation can be rewritten as

$$\frac{\partial m}{\partial t} = J(m - L). \tag{10.13}$$

3. Cullen's stability principle asserts that P_g should be a convex function in order for the system to be physically relevant. Under this assumption, give an interpretation of equation (10.13) in the form of

$$\frac{\partial m}{\partial t} = J \cdot (\mathrm{Id} - \Pi_S)(m), \tag{10.14}$$

where $\Pi_S(m)$ is the Brenier projection of the map m onto the set of measure-preserving maps, as introduced in Chapter 3. Give a sufficient condition for this map to be well-defined at m.

Equation (10.14) is called a "dynamic rearrangement equation"; and it looks like a differential equation of the form $\dot{m} = F(m)$. It now becomes more intuitive why we should expect the semi-geostrophic system to be well-posed!

4. We now turn to **dual variables**. Introduce the Legendre transform P_g^* of P_g and

$$\alpha = \det D^2 P_g^*.$$

We denote by 1 the Lebesgue measure on Ω; show that $\alpha = m\#1$. Show that α solves the partial differential equation

$$\text{(10.15)} \qquad \begin{cases} \dfrac{\partial \alpha}{\partial t} + \nabla \cdot (\alpha W) = 0, \\[2ex] W = J \cdot [\mathrm{Id} - \nabla P_g^*]. \end{cases}$$

Note that the third equation can in fact be dropped, so that the matrix J may be replaced by the usual 2×2 Hamiltonian matrix.

5. Using the notation of subsection 8.3.2, formally rewrite this equation as

$$\frac{\partial \alpha}{\partial t} = \widetilde{J} \operatorname{grad}_W H(\alpha),$$

where the Hamiltonian function H is given by

$$H(\alpha) = \frac{1}{2} W(\alpha, 1)^2.$$

Related rewritings of the semi-geostrophic equations have been crucial to help intuition in the rigorous study of this system, as performed in [**38, 106, 107, 214**] for instance. It is interesting to note that the "linearization" of the Hamiltonian function, according to Section 7.6, turns the incompressible semi-geostrophic equations into the incompressible two-dimensional Euler equation!

Hints: *Question* 1: Note that $Dx/Dt = u$, etc. *Question* 3: Recall that the Brenier projection is given by the polar factorization theorem. *Question* 4: Note that $m(t, x)$ satifies the ordinary differential equation

$$dm/dt = J(m - \nabla P_g^*(t, m)),$$

so we can write down the equation satisfied by $m\#1$. Write down the Monge-Ampère equation associated with the push-forward ∇P_g^*, and note that $L\#1 = 1$.

Remark 10.5. The work by Cullen and Maroofi [**107**] shows that in the compressible case, the Hamiltonian function should contain an additional term like $\int_\Omega \rho^\gamma(x)\,dx$ for some exponent $\gamma > 1$ which is dictated by thermodynamics. This can still be put in the formalism, thanks to the study of displacement convexity performed in Chapter 5. See [**106**] for another example in which the Hamiltonian function has an additional term.

Problem 10. Estimates of smoothing errors

This problem will use some material from Chapters 7 and 9. Its goal is to give some theoretical estimates of smoothing error for numerical methods based on particle simulation, in a framework which is well-adapted to statistical mechanics equations.

Assume we have access to the positions of a certain number of random particles in $\mathbb{R}^d$, whose law is supposed to be the approximation of a probability measure μ of interest. If the particles are indistinguishable, this information about the positions is summarized by the **empirical measure** (or empirical law),

$$\hat{\mu}^N = \frac{1}{N}\sum_{n=1}^{N} \delta_{X_n},$$

which is a random probability measure. Suppose that the empirical measure associated with our particle system is a good approximation of μ, with very high probability. More precisely,

$$\mathbf{P}[W_p(\hat{\mu}^N, \mu) \geq \varepsilon] \leq \eta_p(N, \varepsilon),$$

where $\eta_p(N,\varepsilon)$ is a known function of N and ε, and $\mathbf{P}$ is the probability measure on the probability space. In some of the problems to come, we shall encounter situations in which such information can be obtained.

Suppose we know that μ is smooth and we wish to reconstruct a very good approximation of it (say, in the total variation norm). A standard procedure is to plot the function

$$\hat{\mu}^N_{\zeta_\alpha} = \frac{1}{N}\sum_{n=1}^{N} \zeta_\alpha(x - X_n),$$

where $(\zeta_\alpha)_{\alpha>0}$ is an approximation of the identity in the sense of convolution,

$$\zeta_\alpha(x) = \frac{1}{\alpha^d}\,\zeta\left(\frac{x}{\alpha}\right),$$

$\alpha > 0$, $\mathrm{Supp}(\zeta) \subset B(0,1)$, $\zeta \geq 0$, $\int \zeta = 1$. In order to get relevant estimates, it is important that α be chosen neither too small (in which case we would still see the inhomogeneities caused by N being finite) nor too large (in which case we would wipe out all the details of μ). We shall make this more precise in the sequel, by using some of the tools discussed in these notes (we will not try to look for optimal estimates). The symbols C and C' will stand for various constants independent of N and α, which may change from question to question.

1. Check that

$$\hat{\mu}^N_{\zeta_\alpha} = \hat{\mu}^N * \zeta_\alpha.$$

2. Show that for any $p \geq 1$,

$$W_p(\hat{\mu}^N_{\zeta_\alpha}, \mu * \zeta_\alpha) \leq W_p(\hat{\mu}^N, \mu),$$
$$W_p(\mu * \zeta_\alpha, \mu) \leq C_p\, \alpha,$$

where C_p depends only on p and ζ. Deduce that

$$\alpha \leq \frac{\varepsilon}{2C_p} \quad \Longrightarrow \quad \mathbf{P}[W_p(\hat{\mu}^N_{\zeta_\alpha}, \mu) \geq \varepsilon] \leq \eta_p\left(N, \frac{\varepsilon}{2}\right).$$

3. Now, let us see a more precise estimate based on the properties of the distance W_1. Let f stand for the density of μ, and let

$$\delta(\alpha) = \sup_{|y-x| \leq \alpha} |f(x) - f(y)|$$

be the modulus of continuity of f. Show that there exists a constant C, depending only on ζ, such that

$$\|\hat{\mu}^N_{\zeta_\alpha} - \mu * \zeta_\alpha\|_{L^\infty} \leq \frac{C}{\alpha^{d+1}} W_1(\hat{\mu}^N, \mu),$$
$$\|\mu * \zeta_\alpha - \mu\|_{L^\infty} \leq \delta(\alpha).$$

Deduce that, if f is Lipschitz, then there exists a constant C' such that

$$\|\hat{\mu}^N_{\zeta_\alpha} - f\|_{L^\infty} \leq C'\left(\frac{W_1(\hat{\mu}^N, \mu)}{\alpha^{d+1}} + \alpha\right).$$

In particular, one can choose $\alpha = O(\varepsilon)$ and $K > 0$ in such a way that

$$\mathbf{P}[\|\hat{\mu}^N_{\zeta_\alpha} - f\|_{L^\infty} \geq \varepsilon] \leq \eta_1(N, K\varepsilon^{d+2}).$$

4. Finally, we consider another estimate (a priori more restrictive), based on the material in Chapter 9. Assume that f is Lipschitz and

$$D^2(-\log f) \geq -CI_n, \quad |\nabla(\log f)| \leq C(1 + |x|), \quad \int |x|^2 f(x)\, dx \leq C$$

for some constant $C \in \mathbb{R}$. Show that similar estimates hold true for $f * \zeta_\alpha$, uniformly as $\alpha \to 0$. Show that there exists $C' > 0$ such that, with large probability,

$$I(\hat{\mu}^N * \zeta_\alpha | \mu * \zeta_\alpha) \leq C'(1 + \alpha^{-2}), \tag{10.16}$$

$$W_2(\hat{\mu}^N_{\zeta_\alpha}, \mu * \zeta_\alpha) \leq \varepsilon \quad \Longrightarrow \quad \|\hat{\mu}^N_{\zeta_\alpha} - \mu * \zeta_\alpha\|^2_{TV} \leq C'\Big(\varepsilon(1 + \alpha^{-1}) + \varepsilon^2\Big). \tag{10.17}$$

where $I(\mu|\nu)$ stands for the relative Fisher information, as defined by (9.25). Show that

$$\|\mu * \zeta_\alpha - \mu\|_{TV} \leq C\alpha^{\frac{2}{d+2}}. \tag{10.18}$$

Conclude that we can choose $\alpha = O(\varepsilon^{\frac{d+2}{d+6}})$ and $K > 0$ in such a way that, for ε small enough,

$$\mathbf{P}[\|\hat{\mu}^N_{\zeta_\alpha} - f\|_{TV} \geq \varepsilon^{\frac{2}{d+6}}] \leq \eta_2(N, K\varepsilon).$$

Hints: *Question* 2: The probabilistic interpretation of convolution may help. *Question* 3: Choose $\alpha \leq \varepsilon/(2C')$. *Question* 4: Note that the second derivative of the logarithm of a smooth function g can be bounded in terms of D^2g/g and $|\nabla(\log g)|^2$. To prove (10.16), expand the square norm appearing in the definition of the relative Fisher information as $\int f|\nabla \log(f/g)|^2$, and bound the various terms separately, using the convexity of the Fisher information (9.26); then note that

$$I(\zeta_\alpha) = \alpha^{-2} I(\zeta).$$

To prove (10.17), apply the Csiszár-Kullback-Pinsker inequality (9.46) and the HWI inequality from Theorem 9.17. To prove (10.18), one can separate the integral defining the L^1 norm into two parts: the bulk ($|x| \leq R$, where R will be chosen later) and the tails ($|x| > R$). The bulk can be estimated by using the Lipschitz bound on f, and the tails by using the second order moment bound. Note that better exponents (arbitrarily close to 1/3) can be obtained instead of $2/(d+6)$ if sufficiently many moments of f are assumed to be finite.

Problem 11. Bakry-Émery criterion with skew-symmetric perturbation

This problem requires a very good understanding of the formalism studied in Chapters 8 and 9.

Arnold and Carlen [**19**] proved the following generalization of the Bakry-Émery criterion, i.e. Theorem 9.9 (ii).

Theorem 10.6 (Generalized Bakry-Émery criterion). *Let V be a C^2 function on $\mathbb{R}^n$, satisfying*

$$\frac{V(x)}{|x|} \xrightarrow[|x|\to\infty]{} +\infty, \qquad \int_{\mathbb{R}^n} e^{-V} = 1.$$

Let ξ be a C^1 vector field on $\mathbb{R}^n$, growing at most polynomially at infinity, such that

$$D^2V - \nabla^s \xi \geq \lambda I_n \qquad (\lambda > 0)$$

and

$$\nabla \cdot (\xi e^{-V}) = 0.$$

Then the probability density e^{-V} satisfies a logarithmic Sobolev inequality: for any probability density ρ, one has

$$H(\rho|e^{-V}) \leq \frac{1}{2\lambda} I(\rho|e^{-V}).$$

Here we have used the notation $\nabla^s\xi$ for the symmetrized gradient of ξ, i.e. $(\nabla^s\xi)_{ij} = (\partial\xi_i/\partial x_j + \partial\xi_j/\partial x_i)/2$. Of course the standard Bakry-Émery criterion is just the same, but with $\xi \equiv 0$. The purpose of this problem is to give a simple formal proof of this new result, with the help of Otto's formalism described in Chapter 8. In fact the formal proof can be turned into a rigorous one, by using the rules explained in that chapter.

1. By setting $F(\rho) = H(\rho|e^{-V})$, $A(\rho) = -\nabla \cdot (\rho\xi)$, and using the formalism from Chapter 8, translate Theorem 10.6 into the following abstract result. *Let F be a nonnegative smooth function on a manifold, with $\inf F = 0$, and let A be a vector field satisfying*

$$\operatorname{Hess} F - \operatorname{grad}^S A \geq \lambda \mathrm{Id}\,, \qquad DF \cdot A = 0.$$

Then

$$\|\operatorname{grad} F\|^2 \geq 2\lambda F.$$

Here we have used the notation $\operatorname{grad}^S A$ for the symmetrized gradient of A, which is defined in an abstract way as a quadratic form acting on tangent vectors:

$$\big\langle \operatorname{grad}^S A \cdot \partial_t\rho,\, \partial_t\rho \big\rangle = \big\langle \operatorname{grad} A \cdot \partial_t\rho,\, \partial_t\rho \big\rangle.$$

Moreover, $\operatorname{grad} A$ is defined here via covariant derivatives, in the sense of Riemannian geometry; in particular, it satisfies the rule

$$D\|A\|^2 \cdot (\partial_s\rho) = 2\big\langle \operatorname{grad} A \cdot \partial_s\rho,\, A \big\rangle.$$

2. Introduce the evolution

$$\frac{dX_t}{dt} = -(\operatorname{grad} F + A)(X_t).$$

What is the corresponding partial differential equation?

3. Establish the following abstract identities for the above evolution:

$$\frac{d}{dt} F(X_t) = -\|\operatorname{grad} F(X_t)\|^2,$$

$$\frac{d}{dt}\|\operatorname{grad} F(X_t)\|^2 = -2\Big\langle \big(\operatorname{Hess} F - \operatorname{grad}^S A\big) \cdot \operatorname{grad} F,\, \operatorname{grad} F \Big\rangle.$$

4. Conclude formally with an abstract Bakry-Émery argument.

Hints: *Question* 1: Recall that $\|\operatorname{grad} F\|_W^2 = I(\rho|e^{-V})$. Then recall that $\operatorname{Hess} F \geq \operatorname{Hess} \mathcal{V}$, where $\mathcal{V}(\rho) = \int V\,d\rho$, and that the Hessian of $\mathcal{V}$ is given by

$$\big\langle \operatorname{Hess} \mathcal{V} \cdot \partial_t \rho, \partial_t \rho \big\rangle_\rho = \int \langle D^2 V \cdot v, v\rangle\, d\rho,$$

where $\partial_t \rho = -\nabla \cdot (\rho v)$, $v = \nabla u$. Next check that $DF \cdot A = 0$; for this, differentiate the functional H along the time-evolution $\partial_t \rho = -\nabla \cdot (\rho \xi)$ and use the condition on ξ. The most delicate part is to compute $\operatorname{grad}^S A$; for this use the following rule: if $(\rho_t)_{0 \leq t \leq 1}$ is a geodesic path starting from some tangent vector $\partial_t \rho_0$ at time 0, then

$$\frac{d}{dt}\bigg|_{t=0} \langle A, \partial_t \rho_t\rangle = \big\langle \operatorname{grad}^S A \cdot \partial_t \rho_0, \partial_t \rho_0 \big\rangle.$$

This is a consequence of the rule of parallel transport along geodesics. At this point, note that $\langle A, \partial_t \rho\rangle$ takes the simple form $\int v_t \cdot \xi\, d\rho_t$, and recall the equations of the geodesics in Otto's framework, for instance in the form of the pressureless Euler equations. Conclude that

$$\langle \operatorname{grad}^S A \cdot \partial_t \rho, \partial_t \rho\rangle = \int \langle \nabla^s \xi \cdot v, v\rangle\, d\rho.$$

Question 2: $\partial_t \rho = \Delta \rho + \nabla \cdot (\rho(\nabla V + \xi))$. *Question* 3: Note that $DF \cdot A = 0$ implies $\langle \operatorname{grad} F, A\rangle = 0$, which in turn implies $\operatorname{Hess} F \cdot A + (\operatorname{grad} A)^T \cdot \operatorname{grad} F = 0$ by differentiation along an arbitrary vector field. *Question* 4: Just note that the function F and its first derivative along the evolution are unchanged with respect to the standard Bakry-Émery setting; so the only difference is at the level of the bounds for the second derivative.

Problem 12. The FKG inequalities

This problem is taken from a paper by Caffarelli [**79**]. It is mainly based on Chapter 4.

The so-called FKG inequalities (Fortuin, Kasteleyn and Ginibre) play an important role in statistical mechanics (see [**145, 167**] for early work on the subject). The following version is due to Holley [**167**]. Let Λ be a finite lattice; without loss of generality, Λ is imbedded in $\{0,1\}^N$ for some N, and therefore equipped with the natural partial ordering

$$x \leq y \iff \forall n \in \{1, \dots, N\}, \quad x_n \leq y_n. \tag{10.19}$$

Whenever x and y are two elements of Λ, one can define their infimum $x \wedge y$ and their supremum $x \vee y$. We shall use the shorthand $\mu[x] = \mu[\{x\}]$. Then, one has

Theorem 10.7 (FKG inequalities). *Let μ and ν be two probability measures on Λ, such that*

$$\forall x, y \in \Lambda, \qquad \mu[x \wedge y]\,\nu[x \vee y] \geq \mu[x]\,\nu[y]. \tag{10.20}$$

Then, ν is more concentrated to the right than μ, in the following senses:

(i) there exists $\pi \in \Pi(\mu, \nu)$ such that for $d\pi$-almost all (x, y),

$$x \leq y;$$

(ii) whenever h is a nondecreasing function on Λ, then

$$\int_\Lambda h\, d\mu \leq \int_\Lambda h\, d\nu.$$

0. Prove that statement (i) implies statement (ii).

In a striking paper [**79**], Caffarelli establishes several links between these inequalities and the Monge-Kantorovich problem. In particular, he states and proves a continuous version of Holley's theorem, in which the optimal transference plan π satisfies statement (i), and he is able to recover the discrete case mentioned above as a limit of this continuous variant.

In this problem, we shall present Caffarelli's proof in the particular case when $\Lambda = \{0, 1\}^N$. Let Q be the unit cube

$$Q = [0, 1]^N,$$

equipped with the partial order (10.19). Let $f(x)$ and $g(y)$ be $C^{1,\alpha}$, positive probability densities on Q. Recall from Problem 7 that there exists a strictly convex function $\varphi \in C^{2,\alpha}(Q)$ such that

$$\nabla\varphi\#(f\, dx) = g\, dy, \qquad \det D^2\varphi(x) = \frac{f(x)}{g(\nabla\varphi(x))} \quad (x \in Q),$$

and moreover $\nabla\varphi$ maps each face of Q onto itself in a $C^{2,\alpha}$ way. From regularity theory for fully nonlinear elliptic equations it can be shown that φ is of class $C^{3,\alpha}$ in the interior of Q and that each directional derivative $\partial\varphi/\partial x_j$ is of class $C^{2,\alpha}$ up to the boundary of Q, out of the faces $(x_j = 0)$ and $(x_j = 1)$. After these preparations, we can state Caffarelli's result:

Theorem 10.8 (FKG inequalities, continuous version). *With the above notation, assume that for all x and y in Q, and for all $j \in \{1, \ldots, N\}$,*

$$x \leq y \text{ and } x_j = y_j \implies \frac{\partial}{\partial x_j}(\log f)(x) \leq \frac{\partial}{\partial y_j}(\log g)(y). \tag{10.21}$$

Then, for all $x \in Q$,

$$\nabla\varphi(x) \geq x. \tag{10.22}$$

In fact, this theorem still holds true under the assumption that f and g are positive bounded functions, if (10.21) is only required to hold in the sense of distributions and (10.22) holds true almost everywhere. This can be shown by a regularization argument. So all the hard work will be to establish Theorem 10.8.

1. Explain why Theorem 10.8 is formally a continuous variant of Theorem 10.7.

2. Establish the following important property on which Caffarelli's argument rests. Let φ be a strictly convex, classical solution of the Monge-Ampère equation

$$\det D^2\varphi(x) = \frac{f(x)}{g(\nabla\varphi(x))}.$$

Then there exists a matrix-valued function $M = (M_{ij})$, symmetric positive, such that for all $k_0 \in \{1, \ldots, N\}$,

$$\sum_{ij} M_{ij} \frac{\partial^2}{\partial x_i x_j}\left(\frac{\partial\varphi}{\partial x_{k_0}}\right) = \frac{\partial}{\partial x_{k_0}}(\log f) - \sum_i \frac{\partial}{\partial x_i}(\log g)(\nabla\varphi)\,\frac{\partial}{\partial x_i}\left(\frac{\partial\varphi}{\partial x_{k_0}}\right).$$

From this deduce that if

$$\zeta_{k_0} \equiv \frac{\partial\varphi}{\partial x_{k_0}} - x_{k_0}$$

has a minimum at $x^0 \in Q$ such that ζ_{k_0} is twice differentiable there, and $x^0_{k_0} \notin \{0, 1\}$, then necessarily

$$\frac{\partial}{\partial x_{k_0}}(\log f)(x^0) \geq \frac{\partial}{\partial x_{k_0}}(\log g)(\nabla\varphi(x^0)).$$

Returning to the proof of Theorem 10.8, let us assume that the inequality in (10.21) is strengthened in the following way: there exist $\delta_0 > 0$ and $\varepsilon > 0$ such that

$$\text{(10.23)} \qquad \forall \delta \leq \delta_0 \quad \forall (x, y) \in Q \times Q \quad \forall j_0 \in \{1, \ldots, N\},$$

$$\left.\begin{array}{r} |y_{j_0} - x_{j_0}| < \delta \\ \forall j \neq j_0, \quad x_j < y_j + \delta \end{array}\right\} \Longrightarrow \quad \frac{\partial}{\partial x_{j_0}}(\log f)(x) \leq \frac{\partial}{\partial y_{j_0}}(\log f)(y) - \varepsilon.$$

This is just a variant of condition (10.21) with strict inequalities here and there. Then define, for any exponent $t \in [0, 1]$,

$$f_t(x) = C(t)\, f(x)^t, \qquad g_t(y) = D(t)\, g(y)^t,$$

where $C(t), D(t)$ are normalizing constants turning f_t and g_t into probability densities. The families $(f_t)_{0 \leq t \leq 1}$ and $(g_t)_{0 \leq t \leq 1}$ define convenient interpolations between the Lebesgue measure and $f_1 = f$ on one hand, the Lebesgue

measure and $g_1 = g$ on the other hand. Let φ_t be convex such that

$$\nabla \varphi_t \# (f_t\, dx) = g_t\, dy.$$

3. Assume that (10.22) is false and deduce that there exists a first $t_0 > 0$ such that the inequality

$$\forall j \in \{1, \ldots, N\}, \quad \forall x \in Q, \qquad \frac{\partial}{\partial x_j} \varphi_t(x) \geq x_j - \frac{\delta}{2}$$

is violated for $t > t_0$ close enough to t_0.

4. Without loss of generality, assume that there exists $x^0 \in Q$ such that

$$\begin{cases} \dfrac{\partial}{\partial x_1} \varphi_{t_0}(x^0) - x_1^0 = -\dfrac{\delta}{2} = \inf_{x \in Q} \left[\dfrac{\partial}{\partial x_1} \varphi_{t_0}(x) - x_1 \right], \\ \\ \forall j \neq 1 \quad \dfrac{\partial}{\partial x_j} \varphi_{t_0}(x^0) \geq x_j^0 - \dfrac{\delta}{2}. \end{cases}$$

Show that $x_1^0 \notin \{0, 1\}$. Use (10.23) and the result of Question 2 to obtain a contradiction.

5. Now we get rid of the extra assumption (10.23). Assume that (10.21) holds true. Replace g by $g_\eta(y) = C_\eta \exp(\eta \sum y_i) g(y)$, where C_η is a normalizing constant, $\eta > 0$, and show that (10.23) is satisfied for g_η in place of g. Use this to conclude the proof of Theorem 10.8.

6. Now let us see how the discrete version of Holley's theorem follows from the continuous one. First note that Λ is the set of vertices of Q. To each $\lambda \in \Lambda$ associate the cube Q_λ defined by

$$Q_\lambda = \left\{ z \in Q; \quad \sup_{1 \leq n \leq N} |x_n - \lambda_n| \leq \frac{1}{2} \right\}.$$

Thus each $x \in Q$ belongs to some Q_λ with $\lambda = \lambda(x)$, unambiguously defined up to a set of zero measure. Let μ and ν be two probability measures satisfying the condition (10.20). Define

$$f = \sum_{\lambda \in \Lambda} \mu[\lambda]\, 1_{Q_\lambda}, \qquad g = \sum_{\lambda \in \Lambda} \nu[\lambda]\, 1_{Q_\lambda}.$$

Show that, in the sense of distributions,

$$\frac{\partial}{\partial x_j} (\log f) = \log \frac{\mu[\lambda_+(x)]}{\mu[\lambda_-(x)]}\, \delta_{x_j = \frac{1}{2}},$$

where $\lambda_+(x)$ (resp. $\lambda_-(x)$) is obtained from $\lambda(x)$ by setting the j-th coordinate equal to 1 (resp. 0); of course $\lambda(x)$ is either $\lambda_+(x)$ or $\lambda_-(x)$. Deduce that f and g satisfy (10.21) in the sense of distributions.

7. Let $\nabla\varphi$ be the Brenier map between f and g, so $\nabla\varphi\#(f\,dx) = g\,dy$. Whenever λ_1 and λ_2 are two elements of Λ, define

$$\pi(\lambda_1, \lambda_2) = \frac{\mu[\lambda_1]}{|Q_{1/2}|}\left|\left\{x \in Q_{\lambda_1}; \quad \nabla\varphi(x) \in Q_{\lambda_2}\right\}\right|,$$

where $Q_{1/2} = [0, 1/2]^N$. Check that

$$\pi(\lambda_1, \lambda_2) = \frac{\nu[\lambda_2]}{|Q_{1/2}|}\left|\left\{y \in Q_{\lambda_2}; \quad \nabla\varphi^*(y) \in Q_{\lambda_1}\right\}\right|.$$

Show that π satisfies the conclusion of point (i) in Theorem 10.7.

Hints: *Question* 1: Replace the derivatives by finite differences. *Question* 2: Recall for instance (3.9). *Question* 3: For $t = 0$ the inequality is true; then apply a continuity argument. *Question* 4: If x_1^0 is equal to 0 or 1, then $\partial_1\varphi_{t_0}(x^0)$ is also equal to 0 or 1. *Question* 6: Note that if $x \le y$ and $x_j = y_j$, then $\lambda(x) \le \lambda(y)$ and

$$\lambda_+(x) \vee \lambda_-(y) = \lambda_+(y), \qquad \lambda_+(x) \wedge \lambda_-(y) = \lambda_-(x).$$

Question 7: Use the definition of push-forward.

Problem 13. Caffarelli's log concave perturbation theorem

The goal of this problem is to get a hint of the way Caffarelli [**79**] proves Theorem 9.14; it is the sequel of Problem 12, but does not use it (it is actually somewhat simpler).

1. Start again from a smooth solution of the Monge-Ampère equation

$$\det D^2\varphi(x) = \frac{f(x)}{g(\nabla\varphi(x))}.$$

Whenever e is a unit vector in $\mathbb{R}^n$, define

$$\partial_e\varphi = \nabla\varphi \cdot e, \qquad \partial_{ee}\varphi = \langle D^2\varphi \cdot e, e\rangle,$$

and establish the identity

$$\begin{aligned}\sum_{ij} M_{ij}\partial_{ij}\partial_{ee}\varphi &+ \sum_{ijk\ell} M'_{ijk\ell}\partial_{ij}\partial_e\varphi\,\partial_{k\ell}\partial_e\varphi \\ &= \partial_{ee}(\log f) - \sum_{ij}\partial_{ij}(\log g)\partial_i\partial_e\varphi\,\partial_j\partial_e\varphi - \sum_i \partial_i(\log g)\,\partial_i\partial_{ee}\varphi.\end{aligned}$$

Deduce from the log concavity of the determinant that the second term in the left-hand side is always nonpositive.

2. Assume that $\partial_{ee}\varphi$ reaches its maximum at some point x^0, and show that necessarily

$$\partial_{ee}(\log f) - \sum_{ij} \partial_{ij}(\log g)\partial_i\partial_e\varphi\, \partial_j\partial_e\varphi \leq 0$$

at x^0.

3. Assume now that f is the standard Gaussian γ on $\mathbb{R}^n$, and that $g(x) = e^{-v(x)}\gamma(x)$ for some convex function v. Show that $\partial_{ee}\varphi \leq 1$.

This makes it quite likely that the supremum of $\partial_{ee}\varphi$, taken on all directions e and on all $\mathbb{R}^n$, should always be less than or equal to 1, which amounts precisely to the statement in Theorem 9.14.

In fact the proof is not so simple, because it is not clear that the supremum of $\partial_{ee}\varphi$ will be attained, and one has to take good care of the behavior of φ at infinity. To circumvent this difficulty, Caffarelli works with $\langle D^2\varphi(x)\cdot e, e\rangle$ replaced by the "finite difference"

$$\varphi(x+he) + \varphi(x-he) - 2\varphi(x),$$

for some fixed $h > 0$. Then several technical steps follow.

As we mentioned in Chapter 9, this theorem implies the powerful Theorem 9.13. Other implications are drawn in [**79**], like the following. Let Γ be a convex function of one variable in $\mathbb{R}^n$. Let v be convex and such that $\mu = e^{-v}\gamma$ is a probability measure. Then,

$$\int_{\mathbb{R}^n} \Gamma\left(x_1 - \int_{\mathbb{R}^n} x_1\, d\mu(x)\right) d\mu(x) \leq \int_{\mathbb{R}^n} \Gamma(x_1)\, d\gamma(x).$$

These inequalities were first proven by Brascamp and Lieb in the case $\Gamma(x_1) = |x_1|^\alpha$.

Hints: *Question* 2: $\partial_i(\partial_{ee}\varphi)$ should be 0 at x^0, and the matrix $D^2\partial_{ee}\varphi$ should be nonpositive. *Question* 3: One finds that $|\nabla\partial_e\varphi|^2 \leq 1$.

Problem 14. Contraction properties of Vlasov equations

This problem rests on the characteristics method for solving transport equations (Theorem 5.34), on the Kantorovich-Rubinstein duality (Theorem 1.14), and on the material in Chapter 7. The goal here is to establish quantitative weak stability estimates for the nonlinear Vlasov equation.

The Vlasov equation is one of the fundamental equations in the theory of interacting particle systems. It is the macroscopic equation associated with the Newton equations

$$\frac{d^2 X_t^i}{dt^2} = \frac{1}{N}\sum_{j=1}^{N} F(X_t^i - X_t^j), \qquad 1 \le i \le N, \tag{10.24}$$

where X_t^i in $\mathbb{R}^d$ stands for the position of particle number i at time t, and $F(x-y)$ is the force exerted at point x by a particle located at point y. The physical units have been chosen in such a way that the normalizing factor $1/N$ appears in front of the right-hand side of (10.24).

To apply Theorem 5.34 to this setting, we should first reduce the system (10.24) to a differential system of first order. For that we introduce the phase space made up of all positions *and velocities* of particles; so the density of particles will be a time-dependent probability measure $d\mu_t(x,v)$ on $\mathbb{R}^d \times \mathbb{R}^d$, with x standing for position and v for velocity. The resulting equation is the Vlasov equation, one of the most popular mean-field equations in mathematical physics:

$$\frac{\partial \mu_t}{\partial t} + v \cdot \nabla_x \mu_t + \overline{F}(t,x) \cdot \nabla_v \mu_t = 0, \tag{10.25}$$

where

$$v \cdot \nabla_x = \sum_{j=1}^{d} v_j \frac{\partial}{\partial x_j},$$

and the force field $\overline{F}$ is coupled to μ via the formula

$$\overline{F}(t,x) = \int_{\mathbb{R}^d} F(x-y)\, d\mu_t(y,w) = F * \rho_t(x),$$

where ρ_t is the density of particles in physical space, i.e. the marginal of μ over the position space.

A particularly famous case of the Vlasov equation is the so-called Vlasov-Poisson equation, in which $F(x) = -\nabla V(x)$ for $V(x) = 1/(4\pi|x|)$ in dimension $d = 3$. This equation is at the basis of the kinetic theory of plasmas. This case is however definitely excluded by the strong assumptions which we shall impose on the force field F in this problem. Indeed, in all the sequel, we shall assume that F is globally Lipschitz on $\mathbb{R}^d$:

$$\|\nabla F\|_{L^\infty} \equiv \sup_{x\in\mathbb{R}^d} \|\nabla F(x)\| < +\infty,$$

where $\|\nabla F(x)\|$ is the operator norm of the matrix $\nabla F(x)$,

$$\|\nabla F(x)\| = \sup_{h\neq 0} \frac{|\nabla F(x)\cdot h|}{|h|}.$$

Of course this assumption ensures that the Newton equations (10.24) can be solved for all times, and define a Lipschitz flow.

0. Let $(X_t^i)_{t\geq 0}$, $1 \leq i \leq N$, be a solution of the Newton equations (10.24), and let

$$\hat{\mu}_t^N = \frac{1}{N}\sum_{i=1}^N \delta_{(X_t^i, V_t^i)}, \qquad V_t^i = \frac{dX_t^i}{dt}.$$

Show that $\hat{\mu}_t^N$ does solve the Vlasov equation (10.25) in the weak sense.

Let us consider the statistical mechanics regime, where N is really huge (say 10^{23}). It would be nonsense to write down the Newton system in this case, or use it in numerical simulations; but we can hope to have an excellent approximation of the behavior of the system by searching for an approximate equation at the level of (10.25), or by performing numerical simulations with a much lower number of particles (say 10^6). For instance, if we know that $\hat{\mu}_0^N$ is very close to some given measure μ_0, is it true that $\hat{\mu}_t^N$ will be very close to some other measure μ_t, deduced from μ_0 via resolution of a partial differential equation, if N is large enough? Of course, we expect this guess to be true, at least in a reasonable range of physical parameters, if $(\mu_t)_{t\geq 0}$ itself solves the Vlasov equation. And if we still have good error estimates for $N = 10^6$, then we know that both the system made up of a very large number of particles, and the one made up of a much smaller number, will have approximately the same behavior. Thus one is naturally led to the problem of finding **stability estimates** for measure solutions to the Vlasov equation. If we want these estimates to hold true for singular probability measures, this forbids the use of Lebesgue spaces, etc. On the other hand, the tools of mass transportation may be very efficient.

In this problem we shall establish such an estimate, going back to the seventies, in particular to independent work by Braun and Hepp, by Dobrushin and by Neunzert. Precise references can be found in [**240**, chapter 5].

Theorem 10.9 (A stability estimate for the Vlasov equation). *Let μ_t and ν_t be two solutions of the Vlasov equation* (10.25)*, with respective initial data $\mu_0, \nu_0 \in P_1(\mathbb{R}^d \times \mathbb{R}^d)$. Then*

$$\forall t \geq 0, \quad W_1(\mu_t, \nu_t) \leq e^{2Lt} W_1(\mu_0, \nu_0), \tag{10.26}$$

where $L = \max(\|\nabla F\|_{L^\infty}, 1)$.

Here it will be useful to work with the W_1 distance, so as to use the Kantorovich-Rubinstein representation. We however mention that a stronger estimate can be proven via the use of the W_2 distance:

$$W_2(\mu_t, \nu_t) \leq e^{Lt} W_2(\mu_0, \nu_0).$$

The reader may try to prove this bound for himself/herself with the help of the material in Chapter 8 (this requires an excellent understanding of the main results there). For the moment we shall only prove Theorem 10.9.

1. Whenever μ is a probability measure on $\mathbb{R}^d \times \mathbb{R}^d$, define the vector field $\xi[\mu]$ in $\mathbb{R}^d \times \mathbb{R}^d$ by

$$\xi[\mu](x,v) = \big(v, F[\mu](x)\big), \qquad F[\mu](x) = \int_{\mathbb{R}^d\times\mathbb{R}^d} F(x-y)\, d\mu(y,w). \tag{10.27}$$

Show that

$$\|\xi[\mu]\|_{\mathrm{Lip}} \le L. \tag{10.28}$$

When $(\mu_t)_{t\ge 0}$ is a time-dependent probability measure on $\mathbb{R}^d \times \mathbb{R}^d$, say Lipschitz in the sense of W_1 distance, define $(T_t[\mu.])$ to be the field of characteristics associated with the time-dependent vector field $\xi_t = \xi[\mu_t]$. Note carefully that T_t does not depend only on μ_t, but on the whole family $(\mu_s)_{0\le s\le t}$. Rewrite the Vlasov equation as

$$\mu_t = T_t[\mu.]\#\mu_0.$$

Deduce from (10.28) that $T_t[\mu.]$ satisfies the bound

$$\|T_t[\mu.]\|_{\mathrm{Lip}} \le e^{Lt}.$$

2. We now consider two solutions $(\mu_t)_{t\ge 0}$ and $(\nu_t)_{t\ge 0}$, and try to establish (10.26). The key points in the proof consist in quantifying the following facts: (i) close solutions generate close vector fields; (ii) initially close probability measures transported by close vector fields stay close to each other. For (i): Consider two probability measures μ and ν on $\mathbb{R}^d \times \mathbb{R}^d$, and show that

$$\big\|\xi[\mu] - \xi[\nu]\big\|_{L^\infty} \le L W_1(\mu,\nu);$$

then deduce that

$$\big\|T_t[\mu.] - T_t[\nu.]\big\|_{L^\infty} \le L\int_0^t e^{L(t-s)} W_1(\mu_s,\nu_s)\, ds.$$

For (ii), establish the general inequalities

$$W_1(T\#\nu_0, \widetilde{T}\#\nu_0) \le \int |T-\widetilde{T}|\, d\nu_0 \le \|T-\widetilde{T}\|_{L^\infty}$$

and

$$W_1(T\#\nu_0, T\#\mu_0) \le \|T\|_{\mathrm{Lip}} W_1(\mu_0,\nu_0);$$

conclude that in the present situation,

$$W_1(T_t[\mu.]\#\mu_0, T_t[\nu.]\#\nu_0) \le \big\|T_t[\mu.] - T_t[\nu.]\big\|_{L^\infty} + e^{Lt} W_1(\mu_0,\nu_0).$$

3. Put estimates (i) and (ii) together and complete the proof of Theorem 10.9.

4. Application: Consider a particle system satisfying the Newton equations

$$\frac{d^2 X_t^i}{dt^2} = a \sum_{i=1}^{N} \widetilde{F}(X_t^i - X_t^j),$$

where a is some physical constant and $\widetilde{F}$ is a non-dimensional vector field satisfying $\|\nabla \widetilde{F}\|_{L^\infty} \leq 1$. Define $\hat{\mu}_t^N = (1/N) \sum \delta_{(X_t^i, V_t^i)}$, where $V_t^i = dX_t^i/dt$. Assume that $W_1(\hat{\mu}_0^N, \mu_0) \leq 0.001$, for some probability measure μ_0, and let $(\mu_t)_{t \geq 0}$ be the associated solution of the Vlasov equation. Find a time interval (depending on the product Na) for which $W_1(\hat{\mu}_t^N, \mu_t) \leq 0.005$.

Hints: *Question* 0: This is a consequence of Theorem 5.34. *Question* 2: For the first part of (i), use equation (10.27) and recall the Kantorovich-Rubinstein theorem from Chapter 1. Then the second part of (i) follows from

$$\left.\frac{d}{dt}\right|^{+} \|T_t[\mu_\cdot] - T_t[\nu_\cdot]\|_{L^\infty} \leq \|\xi[\mu_t]\|_{\mathrm{Lip}} \|T_t[\mu_\cdot] - T_t[\nu_\cdot]\|_{L^\infty} + \|\xi[\mu_t] - \xi[\nu_t]\|_{L^\infty}$$

and a Gronwall lemma. Next, for the first part of (ii), you may either use the transference plan $(T \times \widetilde{T})\#\nu_0$, or the Kantorovich-Rubinstein theorem. A similar choice is given for the second part of (ii). For the last part of (ii), use the end of Question 1. *Question* 3: This is again a Gronwall lemma.

Problem 15. Convergence estimates in a mean-field limit

This problem mainly rests on Chapters 1 and 9; it is inspired by Malrieu's PhD thesis [**197, 198**]. It will assume from the reader some elementary background on stochastic differential equations. As before, $\mathbf{P}$ and $\mathbf{E}$ will stand respectively for probability and expectation.

Consider a random system of N particles, with respective positions $X_t^i \in \mathbb{R}^d$ $(1 \leq i \leq N)$, obeying the system of stochastic differential equations

$$dX_t^i = dB_t^i - \nabla V(X_t^i)\, dt - \frac{1}{N} \sum_{j=1}^{N} \nabla W(X_t^i - X_t^j)\, dt. \tag{10.29}$$

Here V and W are smooth real-valued functions on $\mathbb{R}^d$. The initial positions of the N particles are assumed to be random, independent and identically distributed according to some probability distribution $\mu_0^{\otimes N}$, with $d\mu_0(x) = f_0(x)\, dx$. Moreover, the $(B_t^i)_{t \geq 0}^{1 \leq i \leq N}$ are N independent Brownian motions in

$\mathbb{R}^d$. We shall assume here that V and W are convex, growing polynomially at infinity, and that $\int |x|^p f_0(x)\,dx < +\infty$ for some p large enough.

Under such assumptions, one can establish that the empirical measure at time t has a deterministic limit as $N \to \infty$:

$$\hat{\mu}_t^N = \frac{1}{N}\sum_{i=1}^N \delta_{X_t^i} \xrightarrow[N\to\infty]{} f_t(x)\,dx. \tag{10.30}$$

Here the convergence as $N \to \infty$ is in the sense of convergence in law of random variables, and for the weak topology of measures. Note that the left-hand side is a random measure, while the right-hand side is deterministic; so (10.30) looks like a law of large numbers. The convergence can be restated as follows: for all $\varphi \in C_b(\mathbb{R}^d)$,

$$\mathbf{E}\left|\frac{1}{N}\sum_{i=1}^N \varphi(X_t^i) - \int_{\mathbb{R}^d}\varphi(x)\,f_t(x)\,dx\right| \xrightarrow[N\to\infty]{} 0.$$

Moreover, the limit density $f_t(x) = f(t,x)$ is characterized as the solution of the McKean-Vlasov equation

$$\frac{\partial f}{\partial t} = \frac{1}{2}\Delta f + \nabla\cdot(f\,\nabla V) + \nabla\cdot(f\,\nabla(f * W)). \tag{10.31}$$

As shown in Sznitman [**244**], this property of the empirical measure becoming deterministic in the limit is equivalent to the requirement that the law of the N particles be **chaotic** as $N \to \infty$, which means, roughly speaking, that k particles among N (k fixed, $N \to \infty$) look like independent, identically distributed random variables. This can be made quantitative with a little bit of work, as follows: one can construct an auxiliary system of identically distributed, *independent* random particles $(Y_t^i)_{1\le i\le N}$, such that the law of each Y_t^i coincides with $f_t(x)\,dx$, and

$$\sup_{t\ge 0}\mathbf{E}\,|X_t^i - Y_t^i|^2 \le \frac{C}{N} \tag{10.32}$$

for some constant $C > 0$ independent of t and N.

The problem addressed here is to find more precise estimates about how close the empirical measure is to its limit value as $N \to \infty$. Letting φ be a smooth test function, can one find a bound on how much the average $(1/N)\sum\varphi(X_t^i)$ deviates from its asymptotic value $\int \varphi(x)\,f_t(x)\,dx$? More precisely, given some $\varepsilon > 0$, can one estimate

$$\mathbf{P}\left(\left|\frac{1}{N}\sum_{i=1}^N \varphi(X_t^i) - \int_{\mathbb{R}^d} f_t(x)\varphi(x)\,dx\right| \ge \varepsilon\right)$$

from above? The motivations for this question do come not only from theoretical purposes, but also from numerical simulations and the will to justify the use of particle methods. Of course the bound should not depend too much on the particular form of φ.

By pushing the coupling argument further, one can in fact replace this question by the easier one in which the system is made up of independent particles, whose common law is given by $f_t(x)\,dx$; then the resulting problem can be attacked by means of limit theorems for empirical measures of identically distributed, independent random variables. This point will be evoked in the next problem.

An alternative way to attack this kind of law of large numbers in an infinite-dimensional setting is the theory of **concentration of measure**, as described in a very sketchy way in Chapter 9. The basic observation is the following: Let φ be a Lipschitz function with Lipschitz constant bounded by 1. Then, from (10.32) we deduce that

$$\begin{aligned}&\left|\mathbf{E}\left(\frac{1}{N}\sum_{i=1}^N \varphi(X_t^i)\right) - \int \varphi(x)\, f_t(x)\,dx\right|\\ &= \left|\mathbf{E}\left(\frac{1}{N}\sum_{i=1}^N \varphi(X_t^i)\right) - \mathbf{E}\left(\frac{1}{N}\sum_{i=1}^N \varphi(Y_t^i)\right)\right| \le \sqrt{\frac{C}{N}}.\end{aligned}$$

So, the mean value of the random variable $(1/N)\sum\varphi(X_t^i)$ exhibits the right behavior. So everything boils down to showing that this random variable is actually very close to its mean value. In the sequel, we shall see a way to prove this. We start with some general facts.

1. Let μ satisfy a Talagrand inequality $T(\lambda)$, i.e. for all $f \in L^1(d\mu)$, $f \ge 0$, $\int f\,d\mu = 1$,

$$W_2(f\mu,\mu) \le \sqrt{\frac{2}{\lambda}\int f\log f\,d\mu}. \tag{10.33}$$

Let φ be a Lipschitz function such that $\int \varphi\,d\mu = 0$, $\|\varphi\|_{\mathrm{Lip}} \le 1$. Show that for all $t > 0$,

$$\int \varphi f\,d\mu \le \frac{t}{2\lambda} + \frac{1}{t}\int f\log f\,d\mu.$$

2. Deduce that

$$\sup_f\left(\int \varphi f\,d\mu - \frac{1}{t}\int f\log f\,d\mu\right) \le \frac{t}{2\lambda},$$

where the supremum is taken over all functions $f \in C_0(\mathbb{R}^d)$ such that $f\mu$ is a probability measure. Identify the supremum in the left-hand side and establish that

$$\int e^{t\varphi}\,d\mu \leq e^{\frac{t^2}{2\lambda}}.$$

3. Prove that for all Lipschitz function φ on $\mathbb{R}^d$,

$$\mu\left[\left|\varphi - \int \varphi\,d\mu\right| \geq \varepsilon\right] \leq 2\exp\left(-\frac{\lambda\varepsilon^2}{2\|\varphi\|_{\mathrm{Lip}}^2}\right). \tag{10.34}$$

4. Deduce the following corollary. Let μ satisfy a logarithmic Sobolev inequality with constant λ on $(\mathbb{R}^d)^N$; then,

$$\|\varphi\|_{\mathrm{Lip}} \leq 1 \Longrightarrow$$
$$\mu\left[\left|\frac{1}{N}\sum_{i=1}^N \varphi(x^i) - \int\left(\frac{1}{N}\sum_{i=1}^N \varphi(x^i)\right)d\mu(x)\right| \geq \varepsilon\right] \leq 2\exp\left(-\frac{\lambda N\varepsilon^2}{2}\right).$$

5. Let μ_t be the law of $(X_t^1, \ldots, X_t^N)$. Apply Itô's formula to show that it solves the linear diffusion equation

$$\frac{\partial \mu_t}{\partial t} = \frac{1}{2}\Delta\mu_t + \nabla\cdot(\nabla \mathbf{V}_N\,\mu_t), \qquad t \geq 0, \quad x \in (\mathbb{R}^d)^N, \tag{10.35}$$

where Δ and ∇ act on $\mathcal{D}'(\mathbb{R}^{dN})$, and

$$\mathbf{V}_N(x^1, \ldots x^N) = \sum_{i=1}^N V(x^i) + \frac{1}{2N}\sum_{i,j} W(x^i - x^j).$$

6. Now we assume that V is β-uniformly convex for some $\beta > 0$. Check that $\mathbf{V}_N$ is itself β-uniformly convex. By a nontrivial theorem due to Bakry [**27**], this implies that μ_t satisfies a logarithmic Sobolev inequality with constant $\lambda_t = [e^{-2\beta t}\lambda_0^{-1} + \beta^{-1}(1 - e^{-2\beta t})]^{-1}$, if μ_0 itself satisfies a logarithmic Sobolev inequality with constant λ_0. Deduce in the end the following estimate:

$$\mathbf{P}\left(\left|\frac{1}{N}\sum_{i=1}^N \varphi(X_t^i) - \int_{\mathbb{R}^d}\varphi(x)\,f_t(x)\,dx\right| \geq \sqrt{\frac{C}{N}} + r\right) \leq 2\exp\left(-\frac{\lambda_t N r^2}{2}\right). \tag{10.36}$$

Hints: *Question* 1: Recall that $W_1 \leq W_2$ and apply the Kantorovich-Rubinstein theorem. Then apply the Talagrand inequality, and conclude with the Young inequality $ab \leq a^2/2 + b^2/2$. *Question* 2: The supremum

is $(1/t)\log(\int e^{t\varphi}\,d\mu)$. *Question* 3: Apply the inequality in Question 2 after choosing $t = \lambda\varepsilon$ and reducing to the case where $\int \varphi\,d\mu = 0$ and $\|\varphi\|_{\mathrm{Lip}} = 1$; then replace φ by $-\varphi$. *Question* 4: First recall from Chapter 9 that a logarithmic Sobolev inequality implies a Talagrand inequality. Then note that the function

$$\Phi(x^1, \ldots, x^N) = \frac{1}{N}\sum_{i=1}^{N} \varphi(x^i)$$

has Lipschitz norm $\|\varphi\|_{\mathrm{Lip}}/\sqrt{N}$.

Remarks 10.10. (i) These results take important advantage of the uniform convexity of V. They do not hold in the case where, for instance, $V = 0$, even if W is uniformly convex. The problem comes from the fact that the quantity $\sum X_t^i$ is invariant under the action of the interaction potential W, and undergoes only diffusion. As shown by Malrieu [**199**], one can fully cure this problem by projecting the whole system onto the hyperplane $(\sum X^i = 0)$, which amounts to getting rid of this "approximate conservation law".

(ii) The main drawback of (10.36) is that it only considers the errors for one function φ at a time; this would not be sufficient in the context of Problem 10. This improvement can be obtained via the use of limit theorems for empirical measures, in the same style as the one below; but at the price of deteriorating the excellent constants in (10.36).

Problem 16. A quantitative Sanov theorem in Monge-Kantorovich distance

This problem requires some familiarity with probability theory and a good mastering of the definition of mass transportation, as well as some material from Chapters 1, 7 and 9. It is partly inspired by Dembo and Zeitouni [**116**, Exercise 6.2.19]. Its goal is to establish the following deviation theorem for empirical measures:

Theorem 10.11 (Logarithmic Sobolev inequality implies a quantitative Sanov-type bound). *Let $(Y_n)_{n\in\mathbb{N}}$ be a sequence of $\mathbb{R}^d$-valued random variables, independent and having common law ν. Assume that ν is absolutely continuous with respect to Lebesgue measure, and satisfies a logarithmic Sobolev inequality LSI(λ). In particular, it follows [recall Theorem 9.9 (iv)] that it is exponentially integrable: there exists $\alpha > 0$ with*

$$M_\alpha \equiv \int_{\mathbb{R}^d} e^{\alpha|x|^2}\,d\nu(x) < +\infty.$$

Moreover, let

$$\hat{\nu}_N = \frac{1}{N}\sum_{n=1}^{N} \delta_{Y_n}$$

be the empirical measure associated with the Y_n. *Then, for all* $\lambda' < \lambda$ *and* $d' > d$ *there exists an explicit constant* C*, depending only on* λ, λ', d, d', α, M_α*, such that if* $N \geq C\varepsilon^{-(d'+2)}$*, then for all* $\varepsilon > 0$,

$$\mathbf{P}\big[W_2(\hat{\nu}_N, \nu) \geq \varepsilon\big] \leq e^{-N\frac{\lambda'}{2}\varepsilon^2} + e^{-(N\varepsilon^{d'+2})^{1/d}}.$$

Here as before, the notation $\mathbf{P}$ stands for the probability on our probability space. There are many possible variants, more or less complicated, under less stringent assumptions on the law ν; so the methods in this problem are more important than the result itself. Accordingly, we did not try to optimize the constants. We also mention that somewhat simpler estimates on $\mathbf{E}\, W_2(\hat{\nu}^N, \nu)^2$, under weaker assumptions, can be found in [**226**, chapter 11], [**174**, Section 4.9] and [**225**, Section 10.2]. A beautiful application of Sanov's theorem expressed in terms of Wasserstein distance can be found in Schochet [**235**]. Finally, a more thorough study has been performed, after the first edition of this book, in [**52**], together with applications.

In the first six questions, we shall go through a truncation procedure which will enable us to reduce to the compact set $B(0, R)$. Then we shall use the bound from Problem 6 to reduce to small balls in the space of probability measures. In Question 11 we shall see a strengthened version of (a subpart of) Sanov's theorem.

1. Let $B_R = B(0, R)$ be the ball centered at 0, with radius R in $\mathbb{R}^d$. Let ν^R be the probability measure obtained by restriction of ν to B_R:

$$\nu^R = \frac{1_{B_R}\nu}{\nu[B_R]}.$$

Introduce a sequence of independent auxiliary random variables $(Z_n)_{n\in\mathbb{N}}$, independent from the Y_n's, with common law ν^R. Define

$$Y_n^R = \begin{cases} Y_n & \text{if } |Y_n| \leq R, \\ Z_n & \text{if } |Y_n| > R. \end{cases}$$

Compute the law of Y_n^R.

2. Show that for $R \geq 1/\sqrt{\alpha}$,

$$W_2(\nu, \nu_R)^2 \leq 4R^2 e^{-\alpha R^2} \int e^{\alpha|x|^2}\, d\nu(x).$$

3. Introduce

$$\nu_N^R = \frac{1}{N}\sum_{n=1}^N \delta_{Y_n^R}.$$

Show that

$$W_2(\hat{\nu}_N, \hat{\nu}_N^R)^2 \leq \frac{1}{N}\sum_{n=1}^N |Y_n - Z_n|^2 1_{|Y_n|\geq R} \leq \frac{4}{N}\sum_{n=1}^N |Y_n|^2 1_{|Y_n|\geq R},$$

and deduce that for $R \geq 1/\sqrt{\alpha}$,

$$\mathbf{P}\big[W_2(\hat{\nu}_N, \hat{\nu}_N^R) \geq \varepsilon\big] \leq \frac{4R^2 e^{-\alpha R^2}}{\varepsilon^2}\int e^{\alpha|x|^2}\, d\nu(x).$$

4. Let μ be a probability measure on B_R, absolutely continuous with respect to ν. Compute $H(\mu|\nu^R) - H(\mu|\nu)$ in terms of $\nu[B_R]$. By using a Chebyshev inequality, show that for R large enough,

$$\begin{aligned} 0 \geq H(\mu|\nu_R) - H(\mu|\nu) &\geq \log\left(1 - e^{-\alpha R^2}\int e^{\alpha|x|^2}\, d\nu(x)\right) \\ &\geq -2e^{-\alpha R^2}\int e^{-\alpha|x|^2}\, d\nu(x). \end{aligned}$$

Conclude that for all probability measures μ on B_R, as soon as R is large enough,

$$H(\mu|\nu_R) \geq H(\mu|\nu) - Ce^{-\alpha R^2}.$$

5. State an interesting inequality between $H(\mu|\nu)$ and $W_2(\mu,\nu)$. Deduce that for R large enough and $\lambda_1 < \lambda$, $\alpha_1 < \alpha$,

$$H(\mu|\nu_R) \geq \frac{\lambda_1}{2} W_2(\mu, \nu_R)^2 - Ce^{-\alpha_1 R^2}, \tag{10.37}$$

where C is a constant independent of R.

6. Show that for all $\eta \in (0,1)$, $\alpha_1 < \alpha$, and for all $\varepsilon > 0$,

$$\mathbf{P}\big[W_2(\hat{\nu}_N, \nu) \geq \varepsilon\big] \leq \mathbf{P}\left[W_2(\hat{\nu}_N^R, \nu^R) \geq (1-\eta)\varepsilon - Ce^{-\frac{\alpha_1}{2}R^2}\right] + \frac{Ce^{-\alpha_1 R^2}}{\varepsilon^2}, \tag{10.38}$$

where C is independent of R and ε.

To summarize the result of these first six questions, we have used the bounds of exponential integrability to reduce to the case when the reference probability distribution is compactly supported. In the next five questions, we shall obtain estimates under this assumption of compact support.

7. Using Problem 6, show that $P(B_R)$ can be covered by $m(R,r)$ balls of radius r in the W_2 distance, where

$$m(R,r) \le \left(\frac{CR}{r}\right)^{C\left(\frac{R^2}{r}\right)^d},$$

and C is a constant depending only on d.

8. Let $\mathcal{B}$ be a measurable subset of $P(B_R)$ (equipped with the Borel σ-algebra generated by the weak topology). Show that whenever $\varphi \in C_b(\mathbb{R}^d)$,

$$\begin{aligned}\mathbf{P}[\hat{\nu}_N^R \in \mathcal{B}] &\le \left(\mathbf{E}\, e^{N\int \varphi\, d\hat{\nu}_N^R}\right) e^{-N \inf_{\mu\in\mathcal{B}}\left(\int \varphi\, d\mu\right)} \\ &= \exp\left(-N \inf_{\mu\in\mathcal{B}} \left[\int \varphi\, d\mu - \frac{1}{N}\log \mathbf{E}\, e^{N\int \varphi\, d\hat{\nu}_N^R}\right]\right).\end{aligned}$$

9. Check that

$$\frac{1}{N}\log \mathbf{E}\, e^{N\int \varphi\, d\hat{\nu}_N^R} = \log \int e^{\varphi}\, d\nu^R$$

and

$$\sup_{\varphi\in C_0(\mathbb{R}^d)} \left[\int \varphi\, d\mu - \log \int e^{\varphi}\, d\nu^R\right] = H(\mu|\nu^R).$$

10. Apply a minimax theorem (for instance, in the form of Theorem 1.9), to prove that if $\mathcal{B}$ is convex and compact, then

$$\begin{aligned}\mathbf{P}[\hat{\nu}_N^R \in \mathcal{B}] &\le \exp\left(-N \sup_{\varphi\in C_0(\mathbb{R}^d)} \inf_{\mu\in\mathcal{B}} \left[\int \varphi\, d\mu - \log\int e^{\varphi}\, d\nu^R\right]\right) \\ &= \exp\left(-N \inf_{\mu\in\mathcal{B}} H(\mu|\nu^R)\right).\end{aligned}$$

11. Let $\mathcal{A}$ be a measurable subset of $P(B_R)$. Show that for any $\delta > 0$,

$$\mathbf{P}[\hat{\nu}_N^R \in \mathcal{A}] \le m(R,\delta/2) e^{-N \inf_{\mu\in\mathcal{A}_\delta} H(\mu|\nu^R)},$$

where $\mathcal{A}_\delta$ is obtained from $\mathcal{A}$ by δ-thickening in the W_2 metric:

$$\mathcal{A}_r = \left\{\mu \in P(B_R);\quad \exists \mu_a \in \mathcal{A};\quad W_2(\mu,\mu_a) \le \delta\right\}.$$

This bound is one of the many possible quantitative upper bounds in Sanov's theorem — which states that the family of empirical measures $(\hat{\nu}_N^R)_{N\in\mathbb{N}}$ satisfies a large deviations principle with rate functional given by $H(\cdot|\nu^R)$; see [**116**].

12. Show that there exists $C > 0$ such that for any $\delta > 0$, $R \geq C$, $\lambda_1' < \lambda_1$ and $\varepsilon > 0$,

$$\mathbf{P}[W_2(\hat{\nu}_N, \nu) \geq \varepsilon] \leq m\left(R, \frac{\delta}{2}\right) \exp\left(-N\left[\frac{\lambda_1'}{2}\varepsilon^2 - Ce^{-\alpha_1 R^2} - C\delta^2\right]\right) + \frac{Ce^{-\alpha_1 R^2}}{\varepsilon^2}.$$

By a proper choice of δ and R, establish the desired theorem.

Hints: *Question* 1: This is ν^R. *Question* 2: Use the coupling provided by Y_n and Z_n, and the identity

$$|y|^2 1_{|y| \geq R} \leq \frac{e^{\alpha|y|^2}}{e^{\alpha R^2}} R^2.$$

Question 3: Establish a convenient transportation plan. *Question* 4: One finds $\nu[B_R]$. Use the fact that $\log(1-x) \geq -2x$ for x small enough. *Question* 5: Recall that a logarithmic Sobolev inequality implies a Talagrand inequality. Note that for all $\lambda_1 < \lambda$, one can write $\lambda(a-b)^2 \geq \lambda_1 a^2 - Cb^2$. *Question* 6: Use the triangle inequality for W_2. *Question* 9: Make explicit use of the definition of empirical measure. *Question* 10: Note that the compactness of $\mathcal{B}$ implies the lower semi-continuity of its indicator function (which is $\{0, +\infty\}$-valued). *Question* 11: Apply the preceding bound on each ball after showing that it is convex and compact for the weak topology; use the triangle inequality again for W_2. *Question* 12: Choose R^2 of the order of $C(N\varepsilon^{d'+2})^{1/d}$ for C large enough.

Bibliography

[1] ABRAHAMSSON, F. Strong L^1 convergence to equilibrium without entropy conditions for the spatially homogeneous Boltzmann equation. *Comm. Partial Differential Equations 24*, 7-8 (1999), 1501–1535.

[2] ADAMS, R. A. *Sobolev spaces.* Academic Press [A subsidiary of Harcourt Brace Jovanovich, Publishers], New York-London, 1975. Pure and Applied Mathematics, Vol. 65.

[3] AGUEH, M. *Existence of solutions to degenerate parabolic equations via the Monge-Kantorovich theory.* PhD thesis, Georgia Tech, Atlanta, 2001.

[4] ALBERTI, G., AND AMBROSIO, L. A geometrical approach to monotone functions in $\mathbb{R}^n$. *Math. Z. 230*, 2 (1999), 259–316.

[5] ALBERTI, G., AMBROSIO, L., AND CANNARSA, P. On the singularities of convex functions. *Manuscripta Math. 76* (1992), 421–435.

[6] ALEKSANDROV, A. Almost everywhere existence of the second differential of a convex function and some properties of convex surfaces connected to it (in Russian). *Učen. Zap. Leningrad. Gos. Univ. 37* (1939), 3–35.

[7] ALEKSANDROV, A. Existence and uniqueness of a convex surface with given integral curvature. *C. R. (Dokl.) Acad. Sci. URSS 35* (1942), 131–134.

[8] ALEKSANDROV, A. Smoothness of a convex surface of bounded gaussian curvature. *C. R. (Dokl.) Acad. Sci. URSS 36* (1942), 195–199.

[9] ALEKSANDROV, A. Dirichlet's problem for the equation $\mathrm{Det}\|z_{ij}\| = \psi(z_1, \cdots, z, x_1, \cdots, x_n)$ (in Russian). *Vestnik Leningrad. Univ. 13* (1958), 5–24.

[10] ALESKER, S., DAR, S., AND MILMAN, V. A remarkable measure preserving diffeomorphism between two convex bodies in $\mathbb{R}^n$. *Geom. Dedicata 74*, 2 (1999), 201–212.

[11] AMBROSIO, L. Movimenti minimizzanti. *Rendiconti Acc. Naz. Sci., XIX 113*, 1 (1995), 191–246.

[12] AMBROSIO, L. Lecture notes on optimal transport problems. In *Mathematical aspects of evolving interfaces* (2003), vol. 1812 of *Lecture Notes in Mathematics*, Springer-Verlag, Berlin, pp. 1–52. Notes from a CIME summer school in Madeira, 2–9 July 2000.

[13] Ambrosio, L., Gigli, N., and Savaré, G. *Gradient flows in metric spaces and in the space of probability measures.* Lectures in Mathematics ETH Zürich. Birkhäuser Verlag, Basel, 2005.

[14] Ambrosio, L., Mainini, E., and Serfaty, S. Gradient flows of the Chapman-Rubinstein-Schatzman model for signed vortices. *Ann. Inst. H. Poincaré Nonlin. Anal. 28*, 2 (2011), 217–246.

[15] Ambrosio, L., and Pratelli, A. Existence and stability results in the L^1 theory of optimal transportation. In *Optimal transportation and applications (Martina Franca, 2001)* (2003), vol. 1813 of *Lecture Notes in Math.*, Springer-Verlag, Berlin, pp. 123–160.

[16] Ambrosio, L., and Serfaty, S. A gradient flow approach to an evolution problem arising in superconductivity. *Comm. Pure Appl. Math. 61*, 11 (2008), 1495–1539.

[17] Ané, C., Blachère, S., Chafaï, D., Fougères, P., Gentil, I., Malrieu, F., Roberto, C., and Scheffer, G. *Sur les inégalités de Sobolev logarithmiques*, vol. 10 of *Panoramas et Synthèses.* Société Mathématique de France, 2000.

[18] Appell, P. Mémoire sur les déblais et les remblais des systèmes continus ou discontinus. *Mémoires présentés par divers Savants à l'Académie des Sciences de l'Institut de France, Paris 29* (1887), 1–208.

[19] Arnold, A., and Carlen, E. A generalized Bakry-Emery condition for non-symmetric diffusions. In *International Conference on Differential Equations, Vol. 1, 2 (Berlin, 1999).* World Sci. Publishing, River Edge, NJ, 2000, pp. 732–734.

[20] Arnold, A., Markowich, P., Toscani, G., and Unterreiter, A. On logarithmic Sobolev inequalities and the rate of convergence to equilibrium for Fokker-Planck type equations. *Comm. Partial Differential Equations 26*, 1–2 (2001), 43–100.

[21] Arnold, V. I. Sur la géométrie différentielle des groupes de Lie de dimension infinie et ses applications à l'hydrodynamique des fluides parfaits. *Ann. Inst. Fourier (Grenoble) 16*, fasc. 1 (1966), 319–361.

[22] Arnold, V. I., and Khesin, B. A. *Topological methods in hydrodynamics.* Springer-Verlag, New York, 1998.

[23] Aronson, D. G., and Bénilan, P. Régularité des solutions de l'équation des milieux poreux dans $\mathbb{R}^N$. *C. R. Acad. Sci. Paris Sér. A-B 288*, 2 (1979), A103–A105.

[24] Aronsson, G. Extension of functions satisfying Lipschitz conditions. *Ark. Mat. 6* (1967), 551–561.

[25] Aubin, T. Problèmes isopérimétriques et espaces de Sobolev. *J. Differential Geometry 11*, 4 (1976), 573–598.

[26] Bakry, D. L'hypercontractivité et son utilisation en théorie des semigroupes. In *Ecole d'été de Probabilités de Saint-Flour*, no. 1581 in Lecture Notes in Math. Springer, 1994, pp. 1–114.

[27] Bakry, D. On Sobolev and logarithmic Sobolev inequalities for Markov semigroups. In *New trends in stochastic analysis (Charingworth, 1994), Taniguchi symposium* (1997), World Sci. Publishing, Singapore, pp. 43–75.

[28] Bakry, D., and Emery, M. Diffusions hypercontractives. In *Sém. Proba. XIX*, no. 1123 in Lecture Notes in Math. Springer, 1985, pp. 177–206.

[29] Barles, G. *Solutions de viscosité des équations de Hamilton-Jacobi.* Springer-Verlag, Paris, 1994.

[30] Barthe, F. *Inégalités fonctionnelles et géométriques obtenues par transport de mesures.* PhD thesis, Univ. Marne-la-Vallée, 1997.

[31] BARTHE, F. On a reverse form of the Brascamp-Lieb inequality. *Invent. Math. 134*, 2 (1998), 335–361.

[32] BECKNER, W. Inequalities in Fourier analysis. *Ann. of Math. (2) 102*, 1 (1975), 159–182.

[33] BECKNER, W. Sobolev inequalities, the Poisson semigroup, and analysis on the sphere S^n. *Proc. Nat. Acad. Sci. U.S.A. 89*, 11 (1992), 4816–4819.

[34] BECKNER, W. Pitt's inequality and the uncertainty principle. *Proc. Amer. Math. Soc. 123*, 6 (1995), 1897–1905.

[35] BECKNER, W. Geometric asymptotics and the logarithmic Sobolev inequality. *Forum Math. 11*, 1 (1999), 105–137.

[36] BECKNER, W., AND PEARSON, M. On sharp Sobolev embedding and the logarithmic Sobolev inequality. *Bull. London Math. Soc. 30*, 1 (1998), 80–84.

[37] BENAMOU, J.-D. Transformations conservant la mesure, mécanique des fluides incompressibles et modèle semi-géostrophique en météorologie. Master's thesis, Univ. Paris-Dauphine, 1992.

[38] BENAMOU, J.-D., AND BRENIER, Y. Weak existence for the semigeostrophic equations formulated as a coupled Monge-Ampère/transport problem. *SIAM J. Appl. Math. 58*, 5 (1998), 1450–1461.

[39] BENAMOU, J.-D., AND BRENIER, Y. A numerical method for the optimal time-continuous mass transport problem and related problems. In *Monge Ampère equation: applications to geometry and optimization (Deerfield Beach, FL, 1997)*, vol. 226 of *Contemp. Math.* Amer. Math. Soc., Providence, RI, 1999, pp. 1–11.

[40] BENEDETTO, D., CAGLIOTI, E., CARRILLO, J. A., AND PULVIRENTI, M. A non-Maxwellian steady distribution for one-dimensional granular media. *J. Statist. Phys. 91*, 5-6 (1998), 979–990.

[41] BENEDETTO, D., CAGLIOTI, E., AND PULVIRENTI, M. A kinetic equation for granular media. *RAIRO Modél. Math. Anal. Numér. 31*, 5 (1997), 615–641. Erratum in *M2AN Math. Model. Numer. Anal. 33*, 2 (1999), 439–441.

[42] BIANE, P., AND VOICULESCU, D. A free probability analogue of the Wasserstein metric on the trace-state space. *Geom. Funct. Anal. 11*, 6 (2001), 1125–1138.

[43] BILLINGSLEY, P. *Convergence of probability measures*, second ed. John Wiley & Sons Inc., New York, 1999. A Wiley-Interscience Publication.

[44] BLACHMAN, N. The convolution inequality for entropy powers. *IEEE Trans. Inform. Theory 2* (1965), 267–271.

[45] BLOWER, G. The Gaussian isoperimetric inequality and transportation. *Positivity 7*, 3 (2003), 203–224.

[46] BOBKOV, S., GENTIL, I., AND LEDOUX, M. Hypercontractivity of Hamilton-Jacobi equations. *J. Math. Pures Appl. 80*, 7 (2001), 669–696.

[47] BOBKOV, S. G. A functional form of the isoperimetric inequality for the Gaussian measure. *J. Funct. Anal. 135*, 1 (1996), 39–49.

[48] BOBKOV, S. G., AND GÖTZE, F. Exponential integrability and transportation cost related to logarithmic Sobolev inequalities. *J. Funct. Anal. 163*, 1 (1999), 1–28.

[49] BOBKOV, S. G., AND HOUDRÉ, C. Some connections between isoperimetric and Sobolev-type inequalities. *Mem. Amer. Math. Soc. 129*, 616 (1997).

[50] BOBKOV, S. G., AND LEDOUX, M. From Brunn-Minkowski to Brascamp-Lieb and to logarithmic Sobolev inequalities. *Geom. Funct. Anal. 10*, 5 (2000), 1028–1052.

[51] Bolley, F. Autour de l'équation de Boltzmann. Tech. rep., École normale supérieure, Paris, 2001.

[52] Bolley, F., Guillin, A., and Villani, C. Quantitative concentration inequalities for empirical measures on non-compact spaces. *Probab. Theory Related Fields 137*, 3-4 (1007), 287–314.

[53] Bonami, A. Étude des coefficients de Fourier des fonctions de $L^p(G)$. *Ann. Inst. Fourier (Grenoble) 20*, fasc. 2 (1970), 335–402 (1971).

[54] Borell, C. The Brunn-Minkowski inequality in Gauss space. *Invent. Math. 30*, 2 (1975), 207–216.

[55] Bouchut, F., and James, F. Duality solutions for pressureless gases, monotone scalar conservation laws, and uniqueness. *Comm. Partial Differential Equations 24*, 11-12 (1999), 2173–2189.

[56] Brascamp, H. J., and Lieb, E. H. Best constants in Young's inequality, its converse, and its generalization to more than three functions. *Advances in Math. 20*, 2 (1976), 151–173.

[57] Brascamp, H. J., and Lieb, E. H. On extensions of the Brunn-Minkowski and Prékopa-Leindler theorems, including inequalities for log concave functions, and with an application to the diffusion equation. *J. Funct. Anal. 22*, 4 (1976), 366–389.

[58] Brenier, Y. Décomposition polaire et réarrangement monotone des champs de vecteurs. *C.R. Acad. Sci. Paris, Série I, 305* (1987), 805–808.

[59] Brenier, Y. Polar factorization and monotone rearrangement of vector-valued functions. *Comm. Pure Appl. Math. 44*, 4 (1991), 375–417.

[60] Brenier, Y. The dual least action problem for an ideal, incompressible fluid. *Arch. Rational Mech. Anal. 122*, 4 (1993), 323–351.

[61] Brenier, Y. On the motion of an ideal incompressible fluid. In *Partial differential equations of elliptic type (Cortona, 1992)*. Cambridge Univ. Press, Cambridge, 1994, pp. 123–148.

[62] Brenier, Y. Minimal geodesics on groups of volume-preserving maps and generalized solutions of the Euler equations. *Comm. Pure Appl. Math. 52*, 4 (1999), 411–452.

[63] Brenier, Y. Transformations conservant la mesure, hydrodynamique et interaction coulombienne. In *Mathématique et physique*. Soc. Math. France, Paris, 1999, pp. 15–27.

[64] Brenier, Y. Extended Monge-Kantorovich theory. In *Optimal transportation and applications (Martina Franca, 2001)* (2003), vol. 1813 of *Lecture Notes in Math.*, Springer-Verlag, Berlin, pp. 91–121.

[65] Brenier, Y. Extension of the Monge–Kantorovich theory to classical electrodynamics. In *Recent advances in the theory and applications of mass transport* (2004), vol. 353 of *Contemp. Math.*, Amer. Math. Soc., Providence, RI, pp. 19–41. Proceedings of the Summer School "Mass Transportation Methods in Kinetic Theory and Hydrodynamics", Ponta Delgada, Azores, September 2000.

[66] Brenier, Y., and Gangbo, W. L^p approximation of maps by diffeomorphisms. To appear in *Calc. Var. Partial Differential Equations.*

[67] Brenier, Y., and Grenier, E. Sticky particles and scalar conservation laws. *SIAM J. Numer. Anal. 35*, 6 (1998), 2317–2328.

[68] Brézis, H. *Analyse fonctionnelle. Théorie et applications.* Masson, Paris, 1983.

[69] BURAGO, Y. D., AND ZALGALLER, V. A. *Geometric inequalities.* Springer-Verlag, Berlin, 1988. Translated from the Russian by A. B. Sosinskiĭ, Springer Series in Soviet Mathematics.

[70] BURTON, G. R., AND DOUGLAS, R. J. Rearrangements and polar factorisation of countably degenerate functions. *Proc. Roy. Soc. Edinburgh Sect. A 128*, 4 (1998), 671–681.

[71] CAFFARELLI, L. A. Interior $W^{2,p}$ estimates for solutions of the Monge-Ampère equation. *Ann. of Math. (2) 131*, 1 (1990), 135–150.

[72] CAFFARELLI, L. A. A localization property of viscosity solutions to the Monge-Ampère equation and their strict convexity. *Ann. of Math. (2) 131*, 1 (1990), 129–134.

[73] CAFFARELLI, L. A. Some regularity properties of solutions of Monge-Ampère equation. *Comm. Pure Appl. Math. 44*, 8-9 (1991), 965–969.

[74] CAFFARELLI, L. A. Boundary regularity of maps with convex potentials. *Comm. Pure Appl. Math. 45*, 9 (1992), 1141–1151.

[75] CAFFARELLI, L. A. The regularity of mappings with a convex potential. *J. Amer. Math. Soc. 5*, 1 (1992), 99–104.

[76] CAFFARELLI, L. A. A note on the degeneracy of convex solutions to Monge-Ampère equation. *Comm. Partial Differential Equations 18*, 7-8 (1993), 1213–1217.

[77] CAFFARELLI, L. A. Allocation maps with general cost functions. In *Partial differential equations and applications.* Dekker, New York, 1996, pp. 29–35.

[78] CAFFARELLI, L. A. Boundary regularity of maps with convex potentials. II. *Ann. of Math. (2) 144*, 3 (1996), 453–496.

[79] CAFFARELLI, L. A. Monotonicity properties of optimal transportation and the FKG and related inequalities. *Comm. Math. Phys. 214*, 3 (2000), 547–563. Erratum in *Comm. Math. Phys.* 225 (2002), no. 2, 449–450.

[80] CAFFARELLI, L. A., AND CABRÉ, X. *Fully nonlinear elliptic equations.* American Mathematical Society, Providence, RI, 1995.

[81] CAFFARELLI, L. A., FELDMAN, M., AND MCCANN, R. J. Constructing optimal maps for Monge's transport problem as a limit of strictly convex costs. *J. Amer. Math. Soc. 15*, 1 (2002), 1–26.

[82] CAGLIOTI, E., AND VILLANI, C. Homogeneous cooling states are not always good approximations to granular flows. *Arch. Ration. Mech. Anal. 163*, 4 (2002), 329–343.

[83] CARLEN, E. Superadditivity of Fisher's information and logarithmic Sobolev inequalities. *J. Funct. Anal. 101*, 1 (1991), 194–211.

[84] CARLEN, E., AND SOFFER, A. Entropy production by block variable summation and central limit theorems. *Comm. Math. Phys. 140* (1991), 339–371.

[85] CARRILLO, J., MCCANN, R. J., AND VILLANI, C. Kinetic equilibration rates for granular media and related equations: Entropy dissipation and mass transportation estimates. *Revista Matemática Iberoamericana 19* (2003), 1–48.

[86] CARRILLO, J., MCCANN, R. J., AND VILLANI, C. Contractions in the 2-Wasserstein length space and thermalization of granular media. *Arch. Rational Mech. Anal. 179* (2006), 217–263.

[87] CARRILLO, J., AND VAZQUEZ, L. Fine asymptotics for fast diffusion equations. *Comm. Partial Differential Equations 28*, 5-6 (2003), 1023–1056.

[88] CARRILLO, J. A., AND TOSCANI, G. Asymptotic L^1-decay of solutions of the porous medium equation to self-similarity. *Indiana Univ. Math. J. 49*, 1 (2000), 113–142.

[89] Cercignani, C. *The Boltzmann equation and its applications.* Springer-Verlag, New York, 1988.

[90] Cercignani, C., Illner, R., and Pulvirenti, M. *The mathematical theory of dilute gases.* Springer, New York, 1994.

[91] Champion, T., and De Pascale, L. The Monge problem for strictly convex norms in $\mathbb{R}^d$. *J. Eur. Math. Soc. 12*, 6 (2010), 1355–1369.

[92] Champion, T., and De Pascale, L. The Monge problem in $\mathbb{R}^d$. *Duke Math. J. 157*, 3 (2011), 551–572.

[93] Chavel, I. *Riemannian geometry—a modern introduction*, vol. 108 of *Cambridge Tracts in Mathematics.* Cambridge University Press, Cambridge, 1993.

[94] Chemin, J.-Y. *Perfect incompressible fluids.* The Clarendon Press Oxford University Press, New York, 1998. Translated from the 1995 French original by Isabelle Gallagher and Dragos Iftimie.

[95] Chorin, A. J., and Marsden, J. E. *A mathematical introduction to fluid mechanics*, third ed., vol. 4 of *Texts in Applied Mathematics.* Springer-Verlag, New York, 1993.

[96] Cordero-Erausquin, D. *Inégalités géométriques.* PhD thesis, Univ. Marne-la-Vallée, 2000.

[97] Cordero-Erausquin, D. Some applications of mass transport to Gaussian-type inequalities. *Arch. Ration. Mech. Anal. 161*, 3 (2002), 257–269.

[98] Cordero-Erausquin, D. Nonsmooth differential properties of optimal transport. In *Recent Advances in the Theory and Applications of Mass Transport* (2004), M. Carvalho and J. Rodrigues, Eds., vol. 353 of *Contemp. Math.*, AMS, Providence, RI, pp. 61–71.

[99] Cordero-Erausquin, D., Gangbo, W., and Houdré, C. Inequalities for generalized entropy and optimal transportation. In *Recent advances in the theory and applications of mass transport* (2004), vol. 353 of *Contemp. Math.*, Amer. Math. Soc., Providence, RI, pp. 73–94. Proceedings of the Summer School "Mass Transportation Methods in Kinetic Theory and Hydrodynamics", Ponta Delgada, Azores, September 2000.

[100] Cordero-Erausquin, D., McCann, R. J., and Schmuckenschläger, M. A Riemannian interpolation inequality à la Borell, Brascamp and Lieb. *Invent. Math. 146*, 2 (2001), 219–257.

[101] Cordero-Erausquin, D., Nazaret, B., and Villani, C. A mass transportation approach to optimal Sobolev and Gagliardo-Nirenberg inequalities. *Adv. Math. 182*, 2 (2004), 307–332.

[102] Cover, T. M., and Thomas, J. A. *Elements of information theory.* John Wiley & Sons Inc., New York, 1991. A Wiley-Interscience Publication.

[103] Crandall, M. G., Evans, L. C., and Gariepy, R. F. Optimal Lipschitz extensions and the infinity Laplacian. *Calc. Var. Partial Differential Equations 13*, 2 (2001), 123–139.

[104] Csiszar, I. Information-type measures of difference of probability distributions and indirect observations. *Stud. Sci. Math. Hung. 2* (1967), 299–318.

[105] Cullen, M. J. P., and Douglas, R. J. Applications of the Monge-Ampère equation and Monge transport problem to meteorology and oceanography. In *Monge Ampère equation: applications to geometry and optimization (Deerfield Beach, FL, 1997)*, Contemp. Math. Amer. Math. Soc., Providence, RI, 1999, pp. 33–53.

[106] CULLEN, M. J. P., AND GANGBO, W. A variational approach for the 2-dimensional semi-geostrophic shallow water equations. *Arch. Ration. Mech. Anal. 156*, 3 (2001), 241–273.

[107] CULLEN, M. J. P., AND MAROOFI, H. The fully compressible semi-geostrophic system from meteorology. *Arch. Ration. Mech. Anal. 167*, 4 (2003), 309–336.

[108] CULLEN, M. J. P., NORBURY, J., AND PURSER, R. J. Generalised Lagrangian solutions for atmospheric and oceanic flows. *SIAM J. Appl. Math. 51*, 1 (1991), 20–31.

[109] CULLEN, M. J. P., AND PURSER, R. J. Properties of the Lagrangian semigeostrophic equations. *J. Atmospheric Sci. 46*, 17 (1989), 2684–2697.

[110] DAFERMOS, C. M. *Hyperbolic conservation laws in continuum physics*, vol. 325 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 2000.

[111] DARBOUX, G. Prix Bordin (géométrie). *C.R. Acad. Sci. Paris 101* (1885), 1312–1316.

[112] DEL PINO, M., AND DOLBEAULT, J. Generalized Sobolev inequalities and asymptotic behaviour in fast diffusion and porous medium problems. Preprint Univ. Paris IX-Dauphine, CEREMADE, no. 9905, 1999.

[113] DEL PINO, M., AND DOLBEAULT, J. Best constants for Gagliardo-Nirenberg inequalities and applications to nonlinear diffusions. *J. Math. Pures Appl. (9) 81*, 9 (2002), 847–875.

[114] DEL PINO, M., AND DOLBEAULT, J. Nonlinear diffusions and optimal constants in Sobolev type inequalities: asymptotic behaviour of equations involving the p-Laplacian. *C. R. Math. Acad. Sci. Paris 334*, 5 (2002), 365–370.

[115] DEMBO, A., COVER, T., AND THOMAS, J. Information theoretic inequalities. *IEEE Trans. Inform. Theory 37*, 6 (1991), 1501–1518.

[116] DEMBO, A., AND ZEITOUNI, O. *Large deviations techniques and applications*, second ed. Springer-Verlag, New York, 1998.

[117] DESVILLETTES, L., AND VILLANI, C. On the spatially homogeneous Landau equation for hard potentials. II. H-theorem and applications. *Comm. Partial Differential Equations 25*, 1-2 (2000), 261–298.

[118] DESVILLETTES, L., AND VILLANI, C. On the trend to global equilibrium in spatially inhomogeneous entropy-dissipating systems: the linear Fokker-Planck equation. *Comm. Pure Appl. Math. 54*, 1 (2001), 1–42.

[119] DESVILLETTES, L., AND VILLANI, C. On a variant of Korn's inequality arising in statistical mechanics. *ESAIM: Control Optim. Calc. Var. 8* (2002), 603–619.

[120] DJELLOUT, H., GUILLIN, A., AND WU, L. Transportation cost-information inequalities and applications to random dynamical systems and diffusions. *Ann. Probab. 32*, 3B (2004), 2702–2732.

[121] DO CARMO, M. P. *Riemannian geometry*. Mathematics: Theory & Applications. Birkhäuser Boston Inc., Boston, MA, 1992. Translated from the second Portuguese edition by Francis Flaherty.

[122] DRUET, O. Isoperimetric inequalities on compact manifolds. *Geom. Dedicata 90* (2002), 217–236.

[123] DRUET, O. Sharp local isoperimetric inequalities involving the scalar curvature. *Proc. Amer. Math. Soc. 130*, 8 (2002), 2351–2361.

[124] DUDLEY, R. *Probabilities and metrics : Convergence of laws on metric spaces, with a view to statistical testing*. Aarhus Universitet, 1976.

[125] DUDLEY, R. M. Metric marginal problems for set-valued or non-measurable variables. *Probab. Theory Related Fields 100*, 2 (1994), 175–189.

[126] DUDLEY, R. M. *Uniform central limit theorems*, vol. 63 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1999.

[127] DUDLEY, R. M. *Real analysis and probability*, vol. 74 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2002. Revised reprint of the 1989 original.

[128] E, W., RYKOV, Y. G., AND SINAI, Y. G. Generalized variational principles, global weak solutions and behavior with random initial data for systems of conservation laws arising in adhesion particle dynamics. *Comm. Math. Phys. 177*, 2 (1996), 349–380.

[129] EBIN, D. G., AND MARSDEN, J. Groups of diffeomorphisms and the notion of an incompressible fluid. *Ann. of Math. (2) 92* (1970), 102–163.

[130] ELIASSEN, A. The quasi-static equations of motion with pressure as independent variable. *Geofys. Publ. Norske Vid.-Akad. Oslo 17*, 3 (1948).

[131] EVANS, L. Some new PDE methods for weak KAM theory. *Calc. Var. Partial Diff. Eq. 17* (2003), 159–177.

[132] EVANS, L. Three singular variational problems. In *Viscosity solutions of differential equations and related topics* (2004), vol. 1323 of *RIMS Kokyuroku*, Research Institute for the Mathematical Sciences, Kyoto. Available online at `https://math.berkeley.edu/~evans/three.singular.problems.pdf`.

[133] EVANS, L. C. *Partial differential equations*. American Mathematical Society, Providence, RI, 1998.

[134] EVANS, L. C. Partial differential equations and Monge-Kantorovich mass transfer. In *Current developments in mathematics, 1997 (Cambridge, MA)*. Int. Press, Boston, MA, 1999, pp. 65–126.

[135] EVANS, L. C., AND GANGBO, W. Differential equations methods for the Monge-Kantorovich mass transfer problem. *Mem. Amer. Math. Soc. 137*, 653 (1999), viii+66.

[136] EVANS, L. C., AND GARIEPY, R. F. *Measure theory and fine properties of functions*. CRC Press, Boca Raton, FL, 1992.

[137] EVANS, L. C., AND GOMES, D. Effective Hamiltonians and averaging for Hamiltonian dynamics. I. *Arch. Ration. Mech. Anal. 157*, 1 (2001), 1–33.

[138] EVANS, L. C., AND GOMES, D. Effective Hamiltonians and averaging for Hamiltonian dynamics. II. *Arch. Ration. Mech. Anal. 161*, 4 (2002), 271–305.

[139] FATHI, A. *Weak KAM theorem in Lagrangian dynamics*. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 2012.

[140] FIGALLI, A., RIFFORD, L., AND VILLANI, C. On the Ma-Trudinger-Wang curvature on surfaces. *Calc. Var. Partial Differential Equations 39* (2010), 307–332.

[141] FILIPPOV, A. Differential equations with discontinuous righthand side. *A.M.S. Transl. Ser. 2 42* (1964), 199–231.

[142] FILIPPOV, A. *Differential equations with discontinuous righthand side*. Mathematics and its applications. Kluwer Acad., Dordrecht, Boston, London, 1988.

[143] FISHER, R. Theory of statistical estimation. *Proc. Cambridge Philos. Soc. 22* (1925), 700–725.

[144] FONSECA, I., AND MÜLLER, S. A uniqueness proof for the Wulff theorem. *Proc. Roy. Soc. Edinburgh Sect. A 119*, 1-2 (1991), 125–136.

[145] Fortuin, C. M., Kasteleyn, P. W., and Ginibre, J. Correlation inequalities on some partially ordered sets. *Comm. Math. Phys. 22* (1971), 89–103.

[146] Forzani, L., and Maldonado, D. Properties of the solutions to the Monge-Ampère equation. *Nonlinear Anal. 57*, 5–6 (2004), 815–829.

[147] Gallot, S., Hulin, D., and Lafontaine, J. *Riemannian geometry*, second ed. Universitext. Springer-Verlag, Berlin, 1990.

[148] Gangbo, W. An elementary proof of the polar factorization of vector-valued functions. *Arch. Rational Mech. Anal. 128*, 4 (1994), 381–399.

[149] Gangbo, W. The Monge mass transfer problem and its applications. In *Monge Ampère equation: applications to geometry and optimization (Deerfield Beach, FL, 1997)*, vol. 226 of *Contemp. Math.* Amer. Math. Soc., Providence, RI, 1999, pp. 79–104.

[150] Gangbo, W., and McCann, R. J. Optimal maps in Monge's mass transport problem. *C. R. Acad. Sci. Paris Sér. I Math. 321*, 12 (1995), 1653–1658.

[151] Gangbo, W., and McCann, R. J. The geometry of optimal transportation. *Acta Math. 177*, 2 (1996), 113–161.

[152] Gardner, R. The Brunn-Minkowski inequality. *Bull. Amer. Math. Soc. (N.S.) 39*, 3 (2002), 355–405.

[153] Gentil, I., and Malrieu, F. Équations de Hamilton-Jacobi et inégalités entropiques généralisées. *C. R. Math. Acad. Sci. Paris 335*, 5 (2002), 437–440.

[154] Ghys, E. Gaspard Monge – le mémoire sur les déblais et les remblais. Images des Mathématiques (`www.images.math.cnrs.fr`), January 2012.

[155] Gilbarg, D., and Trudinger, N. S. *Elliptic partial differential equations of second order.* Classics in Mathematics. Springer-Verlag, Berlin, 2001. Reprint of the 1998 edition.

[156] Glimm, J., and Jaffe, A. *Quantum physics*, second ed. Springer-Verlag, New York, 1987. A functional integral point of view.

[157] Graf, S., and Mauldin, R. D. A classification of disintegrations of measures. In *Measure and measurable dynamics (Rochester, NY, 1987)*, no. 94 in Contemp. Math. Amer. Math. Soc., Providence, RI, 1989, pp. 147–158.

[158] Grenier, E. Existence globale pour le système de gaz sans pression. *C.R. Acad. Sci. Paris, Série I, 321*, 2 (1995), 171–174.

[159] Gromov, M. Convex sets and Kähler manifolds. In *Advances in differential geometry and topology.* World Sci. Publishing, Teaneck, NJ, 1990, pp. 1–38.

[160] Gross, L. Logarithmic Sobolev inequalities. *Amer. J. Math. 97* (1975), 1061–1083.

[161] Gross, L. Logarithmic Sobolev inequalities and contractivity properties of semigroups. In *Dirichlet forms (Varenna, 1992).* Springer, Berlin, 1993, pp. 54–88. Lecture Notes in Math., 1563.

[162] Guionnet, A., and Zegarlinski, B. *Lectures on logarithmic Sobolev inequalities*, vol. 1801 of *Lecture Notes in Mathematics.* Springer-Verlag, 2003. pp. 1–134.

[163] Guittet, K. *Contributions à la résolution numérique de problèmes de transport optimal de masse.* PhD thesis, Univ. Paris 6, 2003.

[164] Gutiérrez, C. E. *The Monge-Ampère equation*, vol. 44 of *Progress in Nonlinear Differential Equations and their Applications.* Birkhäuser Boston Inc., Boston, MA, 2001.

[165] Hanin, L. G. An extension of the Kantorovich norm. In *Monge Ampère equation: applications to geometry and optimization (Deerfield Beach, FL, 1997)*, vol. 226 of *Contemp. Math.* Amer. Math. Soc., Providence, RI, 1999, pp. 113–130.

[166] Hirschman, Jr., I. I. A note on entropy. *Amer. J. Math. 79* (1957), 152–156.

[167] Holley, R. Remarks on the FKG inequalities. *Comm. Math. Phys. 36* (1974), 227–231.

[168] Holley, R., and Stroock, D. Logarithmic Sobolev inequalities and stochastic Ising models. *J. Statist. Phys. 46*, 5-6 (1987), 1159–1194.

[169] Hoskins, B. Atmospheric frontogenesis models: some solutions. *Quart. J. Roy. Meteor. Soc. 97* (1971), 139–153.

[170] Hoskins, B. The geostrophic momentum approximation and the semi-geostrophic equations. *J. Atmosph. Sciences 32* (1975), 233–242.

[171] Hoskins, B. The mathematical theory of frontogenesis. *Ann. Rev. of Fluid Mech. 14* (1982), 131–151.

[172] Ikenberry, E., and Truesdell, C. On the pressures and the flux of energy in a gas according to Maxwell's kinetic theory. I. *J. Rational Mech. Anal. 5* (1956), 1–54.

[173] Jordan, R., Kinderlehrer, D., and Otto, F. The variational formulation of the Fokker-Planck equation. *SIAM J. Math. Anal. 29*, 1 (1998), 1–17.

[174] Kalashnikov, V. V., and Rachev, S. T. *Mathematical methods for construction of queueing models.* The Wadsworth & Brooks/Cole Operations Research Series. Wadsworth & Brooks/Cole Advanced Books & Software, Pacific Grove, CA, 1990. Translated from the Russian by Larry Cannon.

[175] Kantorovich, L. V. On the translocation of masses. *C. R. (Dokl.) Acad. Sci. URSS 37* (1942), 199–201.

[176] Kantorovich, L. V. On a problem of Monge (in Russian). *Uspekhi Mat. Nauk. 3* (1948), 225–226.

[177] Kinderlehrer, D., and Walkington, N. J. Approximation of parabolic equations using the Wasserstein metric. *M2AN Math. Model. Numer. Anal. 33*, 4 (1999), 837–852.

[178] Lederman, C., and Markowich, P. On fast-diffusion equations with infinite equilibrium entropy and finite equilibrium mass. *Comm. Partial Differential Equations 28*, 1-2 (2001), 301–332.

[179] Ledoux, M. On an integral criterion for hypercontractivity of diffusion semigroups and extremal functions. *J. Funct. Anal. 105*, 2 (1992), 444–465.

[180] Ledoux, M. Inégalités isopérimétriques en analyse et probabilités. *Astérisque*, 216 (1993), Exp. No. 773, 5, 343–375. Séminaire Bourbaki, Vol. 1992/93.

[181] Ledoux, M. Concentration of measure and logarithmic Sobolev inequalities. In *Séminaire de Probabilités, XXXIII*, vol. 1704 of *Lecture Notes in Math.* Springer, Berlin, 1999, pp. 120–216.

[182] Ledoux, M. The geometry of Markov diffusion generators. *Ann. Fac. Sci. Toulouse Math. (6) 9*, 2 (2000), 305–366. Probability theory.

[183] Ledoux, M. *The concentration of measure phenomenon.* American Mathematical Society, Providence, RI, 2001.

[184] Lee, T.-Y., and Yau, H.-T. Logarithmic Sobolev inequality for some models of random walks. *Ann. Probab. 26*, 4 (1998), 1855–1873.

[185] Leindler, L. On a certain converse of Hölder's inequality. In *Linear operators and approximation (Proc. Conf., Oberwolfach, 1971).* Birkhäuser, Basel, 1972, pp. 182–184. Internat. Ser. Numer. Math., Vol. 20.

[186] Lieb, E. H. Proof of an entropy conjecture of Wehrl. *Comm. Math. Phys. 62*, 1 (1978), 35–41.

[187] Lieb, E. H. Thomas-Fermi and related theories of atoms and molecules. *Rev. Mod. Phys. 53* (1981), 603–641.

[188] Lieb, E. H. Sharp constants in the Hardy-Littlewood-Sobolev and related inequalities. *Ann. of Math. (2) 118*, 2 (1983), 349–374.

[189] Lieb, E. H. Gaussian kernels have only Gaussian maximizers. *Invent. Math. 102*, 1 (1990), 179–208.

[190] Lieb, E. H., and Loss, M. *Analysis*, second ed. American Mathematical Society, Providence, RI, 2001.

[191] Lieb, E. H., and Yau, H.-T. The Chandrasekhar theory of stellar collapse as the limit of quantum mechanics. *Comm. Math. Phys. 112*, 1 (1987), 147–174.

[192] Lin, F., and Zhang, P. On the hydrodynamic limit of Ginzburg-Landau vortices. *Discrete Contin. Dynam. Systems 6*, 1 (2000), 121–142.

[193] Lions, P.-L. *Generalized solutions of Hamilton-Jacobi equations.* Pitman (Advanced Publishing Program), Boston, Mass., 1982.

[194] Lions, P.-L. *Mathematical topics in fluid mechanics.* Two volumes. The Clarendon Press Oxford University Press, New York, 1996. Oxford Science Publications.

[195] Loeper, G. On the regularity of solutions of optimal transportation problems. *Acta Math. 202* (2009), 241–283.

[196] Ma, X.-N., Trudinger, N. S., and Wang, X.-J. Regularity of potential functions of the optimal transportation problem. *Arch. Ration. Mech. Anal. 177*, 2 (2005), 151–183.

[197] Malrieu, F. *Inégalités de Sobolev logarithmiques pour des problèmes d'évolution non linéaires.* PhD thesis, Univ. Paul Sabatier, Toulouse, 2001.

[198] Malrieu, F. Logarithmic Sobolev inequalities for some nonlinear PDE's. *Stochastic Process. Appl. 95*, 1 (2001), 109–132.

[199] Malrieu, F. Convergence to equilibrium for granular media equations and their Euler schemes. *Ann. Appl. Probab. 13*, 2 (2003), 540–560.

[200] Marchioro, C., and Pulvirenti, M. *Mathematical theory of incompressible nonviscous fluids.* Springer-Verlag, New York, 1994.

[201] McCann, R. J. *A convexity theory for interacting gases and equilibrium crystals.* PhD thesis, Princeton Univ., 1994.

[202] McCann, R. J. Existence and uniqueness of monotone measure-preserving maps. *Duke Math. J. 80*, 2 (1995), 309–323.

[203] McCann, R. J. A convexity principle for interacting gases. *Adv. Math. 128*, 1 (1997), 153–179.

[204] McCann, R. J. Equilibrium shapes for planar crystals in an external field. *Comm. Math. Phys. 195*, 3 (1998), 699–723.

[205] McCann, R. J. Exact solutions to the transportation problem on the line. *R. Soc. Lond. Proc. Ser. A Math. Phys. Eng. Sci. 455*, 1984 (1999), 1341–1380.

[206] McCann, R. J. Polar factorization of maps on Riemannian manifolds. *Geom. Funct. Anal. 11*, 3 (2001), 589–608.

[207] McNamara, S., and Young, W. Kinetics of a one-dimensional granular medium. *Phys. Fluids A 5*, 1 (1993), 34–45.

[208] Milman, V. D., and Schechtman, G. *Asymptotic theory of finite-dimensional normed spaces.* Springer-Verlag, Berlin, 1986. With an appendix by M. Gromov.

[209] Monge, G. Mémoire sur la théorie des déblais et des remblais. In *Histoire de l'Académie Royale des Sciences de Paris* (1781), pp. 666–704.

[210] MURATA, H., AND TANAKA, H. An inequality for certain functional of multidimensional probability distributions. *Hiroshima Math. J. 4* (1974), 75–81.

[211] NIETO, J., POUPAUD, F., AND SOLER, J. High-field limit for the Vlasov-Poisson-Fokker-Planck system. *Arch. Ration. Mech. Anal. 158*, 1 (2001), 29–59.

[212] ONSAGER, L. Statistical hydrodynamics. *Suppl. Nuovo Cimento 6*, 2 (1949), 279–287.

[213] OSSERMAN, R. The isoperimetric inequality. *Bull. Amer. Math. Soc. 84* (1978), 1182–1238.

[214] OTTO, F. Unpublished notes about the semi-geostrophic system.

[215] OTTO, F. Double degenerate diffusion equations as steepest descent. Preprint Univ. Bonn, 1996.

[216] OTTO, F. Lubrication approximation with prescribed nonzero contact angle. *Comm. Partial Differential Equations 23*, 11-12 (1998), 2077–2164.

[217] OTTO, F. Evolution of microstructure in unstable porous media flow: a relaxational approach. *Comm. Pure Appl. Math. 52*, 7 (1999), 873–915.

[218] OTTO, F. The geometry of dissipative evolution equations: the porous medium equation. *Comm. Partial Differential Equations 26*, 1-2 (2001), 101–174.

[219] OTTO, F., AND VILLANI, C. Generalization of an inequality by Talagrand and links with the logarithmic Sobolev inequality. *J. Funct. Anal. 173*, 2 (2000), 361–400.

[220] OTTO, F., AND VILLANI, C. Comment on [**46**]. *J. Math. Pures Appl. (9) 80*, 7 (2001), 697–700.

[221] PINSKER, M. *Information and Information Stability of Random Variables and Processes.* Holden-Day, San Francisco, 1964.

[222] POGORELOV, A. V. *Monge-Ampère equations of elliptic type.* Noordhoff, Groningen, 1964.

[223] POGORELOV, A. V. The Dirichlet problem for the multidimensional analogue of the Monge-Ampère equation. *Dokl. Akad. Nauk SSSR 201* (1971), 790–793. English transl. in *Soviet Math. Dokl. 12* (1971), 1727–1731.

[224] PRÉKOPA, A. On logarithmic concave measures and functions. *Acta Sci. Math. (Szeged) 34* (1973), 335–343.

[225] RACHEV, S., AND RÜSCHENDORF, L. *Mass Transportation Problems. Vol. I: Theory, Vol. II: Applications.* Probability and its applications. Springer-Verlag, New York, 1998.

[226] RACHEV, S. T. *Probability metrics and the stability of stochastic models.* John Wiley & Sons Ltd., Chichester, 1991.

[227] ROCKAFELLAR, R. T. Characterization of the subdifferentials of convex functions. *Pacific J. Math. 17* (1966), 497–510.

[228] ROCKAFELLAR, R. T. *Convex analysis.* Princeton University Press, Princeton, NJ, 1997. Reprint of the 1970 original, Princeton Paperbacks.

[229] ROESCH, M. *Contribution à la description géométrique des fluides parfaits incompressibles.* PhD thesis, Univ. Paris VI, 1994.

[230] ROTHAUS, O. Diffusion on compact Riemannian manifolds and logarithmic Sobolev inequalities. *J. Funct. Anal. 42* (1981), 102–109.

[231] RUDIN, W. *Real and complex analysis*, third ed. McGraw-Hill Book Co., New York, 1987.

[232] RUDIN, W. *Functional analysis*, second ed. International Series in Pure and Applied Mathematics. McGraw-Hill Inc., New York, 1991.

[233] RÜSCHENDORF, L. On c-optimal random variables. *Statist. Probab. Lett. 27*, 3 (1996), 267–270.

[234] SCHNEIDER, R. *Convex bodies: the Brunn-Minkowski theory.* Cambridge University Press, Cambridge, 1993.

[235] SCHOCHET, S. The point-vortex method for periodic weak solutions of the 2-D Euler equations. *Comm. Pure Appl. Math. 49*, 9 (1996), 911–965.

[236] SCHULZ, F. *Regularity theory for quasilinear elliptic systems and Monge-Ampère equations in two dimensions.* Springer-Verlag, Berlin, 1990.

[237] SEVER, M. An existence theorem in the large for zero-pressure gas dynamics. *Differential Integral Equations 14*, 9 (2001), 1077–1092.

[238] SHNIREL'MAN, A. I. The geometry of the group of diffeomorphisms and the dynamics of an ideal incompressible fluid. *Mat. Sb. (N.S.) 128(170)*, 1 (1985), 82–109, 144. English translation in *Math. USSR – Sb. 56* (1987), 79–105.

[239] SHNIREL'MAN, A. I. Attainable diffeomorphisms. *Geom. Funct. Anal. 3*, 3 (1993), 279–294.

[240] SPOHN, H. *Large scale dynamics of interacting particles.* Texts and Monographs in Physics. Springer-Verlag, Berlin, 1991.

[241] STAM, A. Some inequalities satisfied by the quantities of information of Fisher and Shannon. *Inform. Control 2* (1959), 101–112.

[242] SUDAKOV, V. N. Geometric problems in the theory of infinite-dimensional probability distributions. *Proc. Steklov Inst. Math. 141* (1979), 1–178.

[243] SUDAKOV, V. N., AND TSIREL'SON, B. S. Extremal properties of half-spaces for spherically invariant measures. *Zap. Naučn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) 41* (1974), 14–24, 165. Problems in the theory of probability distributions, II. English transl. in *J. Soviet. Math. 9*, 1 (1978), 9–17.

[244] SZNITMAN, A.-S. Topics in propagation of chaos. In *École d'Été de Probabilités de Saint-Flour XIX—1989*, vol. 1464 of *Lecture Notes in Math.* Springer, Berlin, 1991, pp. 165–251.

[245] TALAGRAND, M. Concentration of measure and isoperimetric inequalities in product spaces. *Inst. Hautes Études Sci. Publ. Math.*, 81 (1995), 73–205.

[246] TALAGRAND, M. New concentration inequalities in product spaces. *Invent. Math. 126*, 3 (1996), 505–563.

[247] TALAGRAND, M. Transportation cost for Gaussian and other product measures. *Geom. Funct. Anal. 6*, 3 (1996), 587–600.

[248] TALENTI, G. Best constants in Sobolev inequality. *Ann. Mat. Pura Appl. (IV) 110* (1976), 353–372.

[249] TANAKA, H. An inequality for a functional of probability distributions and its application to Kac's one-dimensional model of a Maxwellian gas. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete 27* (1973), 47–52.

[250] TANAKA, H. Probabilistic treatment of the Boltzmann equation of Maxwellian molecules. *Z. Wahrsch. Verw. Gebiete 46*, 1 (1978/79), 67–105.

[251] TOSCANI, G., AND VILLANI, C. Probability metrics and uniqueness of the solution to the Boltzmann equation for a Maxwell gas. *J. Statist. Phys. 94*, 3–4 (1999), 619–637.

[252] TOSCANI, G., AND VILLANI, C. Sharp entropy dissipation bounds and explicit rate of trend to equilibrium for the spatially homogeneous Boltzmann equation. *Comm. Math. Phys. 203*, 3 (1999), 667–706.

[253] TOSCANI, G., AND VILLANI, C. On the trend to equilibrium for some dissipative systems with slowly increasing a priori bounds. *J. Statist. Phys. 98*, 5–6 (2000), 1279–1309.

[254] TRUDINGER, N. S., AND WANG, X.-J. On the Monge mass transfer problem. *Calc. Var. PDE 13* (2001), 19–31.

[255] URBAS, J. Mass transfer problems. Lecture notes from a course given in Univ. Bonn, 1997–1998.

[256] URBAS, J. The generalized Dirichlet problem for equations of Monge-Ampère type. *Ann. Inst. H. Poincaré Anal. Non Linéaire 3*, 3 (1986), 209–228.

[257] URBAS, J. Regularity of generalized solutions of Monge-Ampère equations. *Math. Z. 197*, 3 (1988), 365–393.

[258] URBAS, J. On the second boundary value problem for equations of Monge-Ampère type. *J. Reine Angew. Math. 487* (1997), 115–124.

[259] URBAS, J. An interior curvature bound for hypersurfaces of prescribed k-th mean curvature. *J. Reine Angew. Math. 519* (2000), 41–57.

[260] URBAS, J. Monotonicity formulae and curvature equations. Preprint Isaac Newton Institute, Cambridge, `www.newton.ac.uk/preprints/NI01008.pdf`, 2001.

[261] VÁZQUEZ, J. L. An introduction to the mathematical theory of the porous medium equation. In *Shape optimization and free boundaries (Montreal, PQ, 1990)*. Kluwer Acad. Publ., Dordrecht, 1992, pp. 347–389.

[262] VILLANI, C. Contribution à l'étude mathématique des collisions en théorie cinétique. Master's thesis, Univ. Paris-Dauphine, France, 2000.

[263] VILLANI, C. On the trend to equilibrium for kinetic equations. *Markov Process. Related Fields 8*, 2 (2002), 237–250. Inhomogeneous random systems (Cergy-Pontoise, 2001).

[264] VILLANI, C. Cercignani's conjecture is sometimes true and always almost true. *Comm. Math. Phys. 234* (2003), 455–490.

[265] VILLANI, C. Optimal transportation, dissipative PDE's and functional inequalities. In *Optimal transportation and applications (Martina Franca, 2001)* (2003), vol. 1813 of *Lecture Notes in Math.*, Springer-Verlag, Berlin, pp. 53–89.

[266] VILLANI, C. Trend to equilibrium for dissipative equations, functional inequalities and mass transportation. In *Recent advances in the theory and applications of mass transport* (2004), vol. 353 of *Contemp. Math.*, Amer. Math. Soc., Providence, RI, pp. 95–109. Proceedings of the Summer School "Mass Transportation Methods in Kinetic Theory and Hydrodynamics", Ponta Delgada, Azores, September 2000.

[267] VILLANI, C. *Optimal Transport, old and new*, vol. 338 of *Grundlehren der Mathematischen Wissenschaften*. Springer-Verlag, Berlin, 2008.

[268] YAU, H.-T. Logarithmic Sobolev inequality for generalized simple exclusion processes. *Probab. Theory Related Fields 109*, 4 (1997), 507–538.

[269] YAU, H.-T. Asymptotic solutions to dynamics of many-body systems and classical continuum equations. In *Current developments in mathematics, 1998 (Cambridge, MA)*. Int. Press, Somerville, MA, 1999, pp. 155–236.

[270] YOUNG, L. Generalized curves and the existence of an attained absolute minimum in the calculus of variations. *Comptes Rendus Soc. Sci. et Lettres Varsovie 30* (1937), 212–234.

[271] YOUNG, L. Generalized surfaces in the calculus of variations. *Ann. of Math. 43* (1942), 84–103.

[272] YOUNG, L. Generalized surfaces in the calculus of variations II. *Ann. of Math. 43* (1942), 530–544.

[273] ZOLOTAREV, V. M. *Modern theory of summation of random variables.* Modern Probability and Statistics. VSP, Utrecht, 1997.

Table of Short Statements

Index

1781 年，Gaspard Monge 定义了“最优输运”问题（即以可能的最小工作量进行质量转移），并将其应用于工程。1942 年，Leonid Kantorovich 将新生的线性规划用于 Monge 问题，并将其应用于经济。1987 年，Yann Brenier 利用最优输运证明了一个保持映射的度量集上新的规划定理，并将其应用于流体力学。

每一个这样的贡献都标志着一个完整数学理论的开端，它有很多意料不到的分支。当前，研究人员从极其多样化的视角来使用和研究 Monge-Kantorovich 问题，这包括概率论、泛函分析、等周问题、偏微分方程乃至气象学。

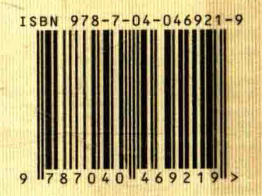
本书源于一门研究生课，可用作最优输运领域的入门书，概述了最近 15 年该领域的研究全貌。本书面向研究生和科研人员，理论和应用并重，读者只需熟悉测度论和泛函分析的基础知识。

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Cédric Villani 的书明晰、可读性极强，它介绍了“最优质量输运”理论的最新进展，及其在最优化理论、非线性偏微分方程、几何、数学物理中各种各样的出乎意料的应用。

—*Lawrence C. Evans*，加州大学伯克利分校


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